AN INFINITESIMAL ANALYSIS OF THE STOP-LOSS-START-GAIN STRATEGY [1]

Siu-Ah Ng

School of Mathematical Sciences, University of Natal

(revised) November 2004, to appear in IJTAF

The paradox of the Stop-Loss-Start-Gain trading strategy is resolved by showing that along the hyperfinite timeline the strategy incurs infinitesimal losses summing up to a non-infinitesimal amount. As a consequence, the Black-Scholes formula is derived using only hyperreal arithmetic and Riemann sum, probably the most elementary derivation thus far.

Key words: Nonstandard analysis, infinitesimal, hyperreal, stop-loss-start-gain, Black-Scholes, asset pricing

1. Introduction

In this paper we apply hyperreal arithmetic and an adoption of the hyperfinite timeline to produce an intuitive and direct resolution of the Stop-Loss-Start-Gain paradox and an explicit computation of the Black-Scholes formula.

The strategy of Stop-Loss-Start-Gain (**SLSG**) is roughly the following: fix a price level K, if the stock price S_t at time t rises from $< K$ to $\geq K$, then buy 1 share immediately (**SG**); if S_t drops from $\geq K$ to $< K$, then sell immediately the share, if any, already hold (**SL**); do nothing in other cases.

Assuming the Black-Scholes economic assumptions such as: no price spread between buy and sell, no restriction on short, trade takes place continuously in time, ... etc., the SLSG produces a portfolio which apparently needs no cost to run, yet, after SG, has a value $(S_T - K)^+$ at any time T, thus replicating the value of an European call option of strike price K expiring at T. Hence there is an apparent contradiction to the Black-Scholes formula which gives a positive value to the European call option. This "paradox" was formally resolved by Carr and Jarrow in [4] (also [3]) by using the local time of Brownian paths. Study of SLSG in other

[1] I am grateful to Bernt Øksendal for suggesting this investigation and would like to thank Peter Carr for his notes [3]. I am supported by South African NRF 2039556.

Address: School of Mathematical Sciences, University of Natal, Pietermaritzburg, 3209 South Africa; e-mail: ngs@nu.ac.za

settings can be found in [7] and [10]. The present article is inspired by Seidenverg's essay [11].[2] We use methods from nonstandard analysis to give a direct but rigorous resolution. In fact we derive the Black-Scholes formula using only hyperreal arithmetic and Riemann sum. Instead of the infinite crossing property of Brownian paths, our approach here is based on following the trading behaviors on the hyperfinite discrete timeline. (*Hyperfinite* means finite in the sense of nonstandard analysis.) An infinitesimal loss is incurred whenever SL is triggered by the discrete movement of the stock price from $S_t \geq K$ to $K > S_{t+\Delta t}$. Then we show that such losses add up to the Black-Scholes price as the time-value of an European call option. Because of the naturalness of our approach, it lends support to the idea that, in a fundamental way, trades take place not continuously but discretely along a hyperfinite timeline. In contrast to the nonstandard derivation of the Black-Schole's formula in either [6] or [9], our technique here is much more elementary: only arithmetic on the hyperreal numbers and basic Riemann sum suffice; neither the Loeb theory nor ω_1-saturation principle is ever needed. The main purpose of the current article is to bring attention to the useful tools of hyperreal arithmetic, although this new derivation may serve as a small dedication to the 30th anniversary of the Black-Scholes formula [2].

In Section 2 we describe some needed background from nonstandard analysis, as well as the assumptions and properties of our model. A more precise formulation of the SLSG strategy is given. Section 3 is the main part of our analysis of the SLSG. The cost of SLSG is calculated and shown to be in agreement with the Black-Scholes price.

2. Preliminaries

Nonstandard analysis.

In a typical setting of Robinson's nonstandard analysis, all mathematical objects X of interest are simultaneously extended to some ${}^*X$, so that they share the same properties definable by first order logic in the language of set theory. Moreover the extension ${}^*X$ is proper when X is infinite.

---

[2]After the first draft of the current paper was written, it was noted that our equation in Proposition 5 is similar to one in [3], but the two approaches diverge afterward.

By an *internal* set we mean elements from some ${}^*X$. We call elements in the set ${}^*\mathbb{N}$ of nonstandard natural numbers *hyperfinite*; a set counted internally by a hyperfinite number is also called hyperfinite; given $r, s \in {}^*\mathbb{R}$, hyperreal numbers, if $|r - s| < q$ for all $q \in \mathbb{R}^+$, we write $r \approx s$ (*infinitely close*); r is called *infinitesimal* when $r \approx 0$; a finite element r of ${}^*\mathbb{R}$ ($r < \infty$) is one with $|r| < n$ for some $n \in \mathbb{N}$; such r is $\approx s$ for a unique $s \in \mathbb{R}$ (called the *standard part*, in symbol: $s = {}^\circ r$); write $r \approx \infty$ when r is infinite. We have the *overspill principle*, *i.e.* if an internal set contains all infinitesimals then it contains a non-infinitesimal hyperreal.[3] Complete background material can be found in [1], [8] or [9].

For simplicity our initial time is 0 and the terminal time is 1. The hyperfinite timeline is defined as

$$\mathbb{T} := \{0, \Delta t, 2\Delta t, \ldots, 1\},$$

where the time increment is $\Delta t = 1/N$ and $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ is a fixed (infinite) hyperfinite number. So each $t \in \mathbb{T}$ has the form $n\Delta t$ for some hyperfinite number n between 0 and N

Hyperfinite CRR model.

Let r denote the riskless interest rate and σ the volatility of the stock return rate. Let K denote some critical price level set at time 1 which, when discounted from time 1, becomes $Ke^{r(t-1)}$ at time t. Define this discounted level as

$$\kappa(t) := Ke^{r(t-1)}.$$

Naturally $r \geq 0$ and σ, $K > 0$ are *standard* real numbers.

Our model for the stock price process is a hyperfinite version of the Cox-Ross-Rubinstein model from [5]. The sample space is $\Omega = \{-1, +1\}^{\mathbb{T}}$ and thus states $\omega = \{\omega_t\}_{t\in\mathbb{T}} \in \Omega$ are sequences of ± 1. We let S be the stock price process, *i.e.* $S : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ and in the state $\omega \in \Omega$ the stock price at time $t \in \mathbb{T}$ is $S_t(\omega)$. (Note that S_0 is a constant.) Under the arbitrage-free assumption (*i.e.* the existence of a unique equivalent martingale measure), S obeys

$$S_{t+\Delta t}(\omega) = S_t(\omega)e^{\omega_t\sigma\sqrt{\Delta t}+r\Delta t}$$

[3] This principle holds in any nontrivial construction of nonstandard objects and is weaker than the ω_1-saturation principle.

and the transition probability $p := \text{Prob}\big(\omega_{t+\Delta t} = 1 \,|\, \omega_s,\, s \le t\big)$ is given by

$$p = \frac{1}{1 + e^{\sigma\sqrt{\Delta t}}}.$$

We remark that since $\sigma > 0$ is finite non-infinitesimal and $\Delta t \approx 0$, it follows that $p < 1 - p$ and $p \approx 1/2$. Now define

$$\pi_t(\omega) \;=\; \sum_{0 \le s < t} \omega_s.$$

(Equivalently, $\pi_t(\omega) \;=\; (\text{number of } +1\text{'s}) - (\text{number of } -1\text{'s}) \quad \text{in} \quad \{\omega_s\}_{0 \le s < t}$.)

So $S_t(\omega) \;=\; S_0\, e^{\pi_t(\omega)\sigma\sqrt{\Delta t}} e^{rt}$.

For $t \,=\, n\Delta t$ it is easy to check that $n - \pi_t(\omega)$ is always an *even number*; in particular, $\pi_t(\omega)$ and n have the same parity.

By choosing S_0 as the numéraire, we can make the simplification that $S_0 = 1$.

The SLSG strategy.

We are now ready to give a precise description of the SLSG strategy and its associated portfolio.

Let $t \in \mathbb{T} \setminus \{1\}$. Then SG is triggered in ω at $t + \Delta t$ when

$$\text{(SG)} \qquad S_t(\omega) < \kappa(t) \quad \text{and} \quad \kappa(t + \Delta t) \le S_{t+\Delta t}(\omega).$$

In this case a share is bought for the portfolio at time $t + \Delta t$ *immediately.*

Similarly SL is triggered in ω at $t + \Delta t$ when

$$\text{(SL)} \qquad S_t(\omega) \ge \kappa(t) \quad \text{and} \quad \kappa(t + \Delta t) > S_{t+\Delta t}(\omega)$$

and a share is sold for the portfolio at time $t + \Delta t$ immediately.

We are only interested in the *time-value* of this strategy hence work in the situation

$$S_0 < \kappa(0), \quad \text{i.e.} \quad 1 < \kappa(0), \quad \text{or} \quad r < \ln K.$$

In particular, SL can only take place after SG.

If SG takes place in ω before time t, we define

$$\tau(t,\omega) \;=\; \min_{0<s<t} \{\, s \,|\, \text{ SG is triggered in } \omega \text{ at } s.\}.$$

If SG is triggered in ω before t, we let $V(t,\omega)$ be the value of holding the share at time t, hence

$$V(t,\omega) \;=\; S_t(\omega) - S_{\tau(t,\omega)}(\omega)\, e^{r(t-\tau(t,\omega))},$$

i.e. the current stock price minus the price paid at SG with interest.[4]

In particular, if SG is triggered in ω before t and no SL is triggered in between, the value of the portfolio produced by the SLSG strategy is simply $V(t,\omega)$; however, if SL is triggered in ω at time $t+\Delta t$, then there is a loss at that time, given by the difference between the interest added value of $V(t,\omega)$ and the value of holding the share, $V(t+\Delta t,\omega)$, *i.e.*

$$\begin{aligned} & V(t,\omega)e^{r\Delta t} - V(t+\Delta t,\omega) \\ = & \Big(S_t(\omega) - S_{\tau(t,\omega)}(\omega)e^{r(t-\tau(t,\omega))}\Big)\, e^{r\Delta t} \; - \; \Big(S_{t+\Delta t}(\omega) - S_{\tau(t,\omega)}(\omega)\, e^{r(t+\Delta t-\tau(t,\omega))}\Big) \\ = & \; S_t(\omega)e^{r\Delta t} - S_{t+\Delta t}(\omega). \end{aligned}$$

Note that the above is infinitesimal and these are the only possible kind of losses incurred in the SLSG strategy. We are interested in the expected infinitesimal loss incurred at time $t+\Delta t$ when SL is triggered and denote it by

$$\mathbb{E}\left[\mathbf{1}_{\left\{S_t(\omega)\geq\kappa(t)\;\&\;\kappa(t+\Delta t)>S_{t+\Delta t}(\omega)\right\}}\left(S_t(\omega)e^{r\Delta t} - S_{t+\Delta t}(\omega)\right)\right],$$

where $\mathbf{1}_C$ is the indicator function of a set C.

Then the expected total loss discounted back to the present is the hyperfinite sum:

$$\sum_{0\leq t<1}\Upsilon(t), \quad \text{where} \tag{1}$$

$$\Upsilon(t) \;:=\; e^{-r(t+\Delta t)}\,\mathbb{E}\left[\mathbf{1}_{\left\{S_t(\omega)\geq\kappa(t)\;\&\;\kappa(t+\Delta t)>S_{t+\Delta t}(\omega)\right\}}\left(S_t(\omega)e^{r\Delta t} - S_{t+\Delta t}(\omega)\right)\right].$$

This hyperfinite sum is of great importance for us and in the next section, we will perform some explicit hyperreal computations and obtain the Black-Scholes formula from it as the time-value of an European call option.

3. Summing up the infinitesimal losses

We say that x and y are *infinitely equivalent* to each other, if $x=\alpha y+\beta$ for some $\alpha\approx 1$ and $\beta\approx 0$. Note that if x and y are infinitely equivalent to each other and one of them is finite, then $x\approx y$. We begin with an analysis of the term $\Upsilon(t)$, then we transform the expected total loss (1) into a sequence of expressions (2) –

[4]Note that $S_{\tau(t,\omega)}(\omega)e^{r(t-\tau(t,\omega))}\approx\kappa(t)$, hence $V(t,\omega)\approx(S_t(\omega)-\kappa(t))^+$, the value of an European call option expired at time t.

(6) and show that they are infinitely equivalent to each other, with the final one (6) equal to the Black-Scholes price.

First an elementary remark often used while dealing with hyperfinite sums.

Lemma 1. *Let I be hyperfinite and suppose for $i \in I$ that $x_i \geq 0$ and $a_i = \lambda_i b_i$, for some $\lambda_i \approx 1$. Then $\sum_{i\in I} a_i x_i = \lambda \sum_{i \in I} b_i x_i$ for some $\lambda \approx 1$.*

Proof. Let $\lambda^+ = \max\{\lambda_i \,|\, i \in I\}$ and $\lambda^- = \min\{\lambda_i \,|\, i \in I\}$ (both ≈ 1), then

$$\lambda^- \sum_{i\in I} b_i x_i \leq \sum_{i \in I} a_i x_i \leq \lambda^+ \sum_{i\in I} b_i x_i \,,$$

so for some λ between λ^+ and λ^- the conclusion holds. □

A characterization of SL.

Lemma 2. *Write $\zeta = \dfrac{\ln K - r}{\sigma\sqrt{\Delta t}}$. (So $\zeta \approx \infty$ by assumptions.) Let $m_\star$ denote the unique integer in the interval $[\zeta, \zeta+1)$. Let $t \in \mathbb{T}$.*

(I) SL is triggered in ω at $t + \Delta t$ iff $\omega_t = -1$ and $\pi_t(\omega) = m_\star$.

(II) Suppose that SL is triggered in ω at $t + \Delta t$. Then

(i) $^\circ\big(m_\star\sqrt{\Delta t}\big) = \dfrac{\ln K - r}{\sigma} > 0$ *and* $m_\star \Delta t \approx 0$.

(ii) *If* $\sqrt{N} t \approx \infty$, *then* $\dfrac{m_\star}{Nt} \approx 0$ *and* $Nt - m_\star \approx \infty$.

Proof. (I): By definition, SL is triggered in ω at $t + \Delta t$ iff $\omega_t = -1$,

$$S_t(\omega) \geq \kappa(t) \quad \text{and} \quad \kappa(t)e^{r\Delta t} = \kappa(t + \Delta t) > S_t(\omega)e^{-\sigma\sqrt{\Delta t} + r\Delta t},$$

i.e.

$$(*) \qquad e^{\sigma\sqrt{\Delta t}} > \frac{S_t(\omega)}{\kappa(t)} \geq 1.$$

Since $S_t(\omega) = e^{\pi_t(\omega)\sigma\sqrt{\Delta t} + rt}$, we have

$$\sigma\sqrt{\Delta t} > \pi_t(\omega)\sigma\sqrt{\Delta t} - \ln K + r \geq 0,$$

equivalently

$$\frac{\ln K - r}{\sigma\sqrt{\Delta t}} \leq \pi_t(\omega) < \frac{\ln K - r}{\sigma\sqrt{\Delta t}} + 1,$$

i.e. $\pi_t(\omega)$ equals the unique integer $m_\star$ from $[\zeta, \zeta + 1)$.

(II): (i) follows from the above inequality.

For (ii), we have $\dfrac{m_\star}{Nt} = \dfrac{m_\star\sqrt{\Delta t}}{\sqrt{N}\, t} \approx 0$ and $Nt - m_\star = Nt\Big(1 - \dfrac{m_\star}{Nt}\Big) \approx \infty$.

□

Remark 3. *We say that t is an* even time *if it has the form $2n\Delta t$ and an* odd time *if it has the form $(2n+1)\Delta t$. Note that $\pi_t(\omega)$ must be even when t is even and odd when t is odd. Consequently, if $m_\star$ is even, then SL is triggered in some ω at $t+\Delta t$ iff t is an even time $\geq m_\star\Delta t$, and similarly if $m_\star$ is odd, then SL is triggered in some ω at $t+\Delta t$ iff t is an odd time $\geq m_\star\Delta t$.*

Now fix any infinite M such that $M\Delta t \approx 0$ but $\sqrt{N}M\Delta t \approx \infty$. (For example, take $M = \lceil N^{3/4} \rceil$.) Note that $M \geq m_\star$.

If $m_\star$ is even, define

$$\mathbb{S} := \{t \in \mathbb{T} \,|\, M\Delta t \leq t < 1 \quad \& \quad t \text{ is even}\},$$

otherwise define

$$\mathbb{S} := \{t \in \mathbb{T} \,|\, M\Delta t \leq t < 1 \quad \& \quad t \text{ is odd}\}.$$

By Lemma 2, Remark 3 and the above, we have

Proposition 4. *Let $t \in \mathbb{T}$ and $t \geq M\Delta t$.*

Then $t \in \mathbb{S}$ iff SL is triggered in some ω at $t+\Delta t$. □

Decomposition of the $\Upsilon(t)$.

Proposition 5. *Let $t = n\Delta t$ and $n \geq m > 0$ of the same parity. Then*

$$\text{Prob}\big(\{\pi_t(\omega) = m\}\big) \;=\; \binom{n}{\frac{n+m}{2}} p^{\frac{n+m}{2}} (1-p)^{\frac{n-m}{2}} \;\leq\; \frac{2}{m}.$$

Proof. We denote $P_i \;=\; \text{Prob}\big(\{\pi_t(\omega) = m-2i\}\big)$, $i = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor$.

Consider a state ω where $\pi_t(\omega) = m - 2i$: $\{\omega_s\}_{0\leq s<t}$ must contain $\dfrac{n+m-2i}{2}$ many $+1$'s and $\dfrac{n-m+2i}{2}$ many -1's. Hence there are precisely $\binom{n}{\frac{n+m-2i}{2}}$ many such sequences $\{\omega_s\}_{0\leq s<t}$. Therefore

$$P_i \;=\; \binom{n}{\frac{n+m}{2} - i} p^{\frac{n+m}{2} - i} (1-p)^{\frac{n-m}{2} + i}.$$

Since $1-p \,>\, p$, we have

$$P_i \;\geq\; \binom{n}{\frac{n+m}{2} - i} p^{\frac{n+m}{2}} (1-p)^{\frac{n-m}{2}}.$$

Moreover

$$\binom{n}{\frac{n+m}{2} - i} \geq \binom{n}{\frac{n+m}{2}}, \qquad i = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor,$$

hence $1 \geq P_i \geq P_0$, for all $i = 0, 1, \dots, \lfloor \frac{m}{2} \rfloor$ and thus $\left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) P_0 \leq 1$. □

Proposition 6. *Let SL be triggered in some* ω *at* $t + \Delta t$*, then for some* $\lambda_t \approx 1$,

$$\Upsilon(t) \ = \ \lambda_t \, \frac{K\sigma}{2e^r} \, \mathrm{Prob}\big(\{\pi_t(\omega) = m_\star\}\big) \, \sqrt{\Delta t}.$$

(Note that if no SL is triggered in any ω *at* $t + \Delta t$*, then* $\Upsilon(t) = 0$*.)*

Proof. Suppose SL is triggered in some ω at $t + \Delta t$. Note that these states ω have identical prices S_t and $S_{t+\Delta t}$. By (*) in the proof of Lemma 2, $S_t(\omega) = \lambda'_t \kappa(t)$ for some $\lambda'_t \approx 1$, independent of any such particular ω.

$$\begin{aligned} S_t(\omega) e^{r\Delta t} - S_{t+\Delta t}(\omega) &= S_t(\omega) e^{r\Delta t} (1 - e^{-\sigma\sqrt{\Delta t}}) \\ &= \lambda'_t \, e^{r\Delta t} \, \kappa(t) (\sigma \sqrt{\Delta t} + \xi_t \Delta t) \end{aligned}$$

for some finite ξ_t, and hence

$$S_t(\omega) e^{r\Delta t} - S_{t+\Delta t}(\omega) \ = \ \lambda''_t \, \kappa(t) \sigma \sqrt{\Delta t}$$

for some $\lambda''_t \approx 1$.

By Lemma 2 and the above, we can write

$$\begin{aligned} \Upsilon(t) \ &= \ e^{-r(t+\Delta t)} \, \mathbb{E}\left[\mathbf{1}_{\{\, \pi_t(\omega) = m_\star \,\&\, \omega_t = -1\}} \big(S_t(\omega) e^{r\Delta t} - S_{t+\Delta t}(\omega) \big) \right] \\ &= \ e^{-r(t+\Delta t)} \, (1-p) \, \mathbb{E}\left[\mathbf{1}_{\{\, \pi_t(\omega) = m_\star\}} \big(S_t(\omega) e^{r\Delta t} - S_{t+\Delta t}(\omega) \big) \right] \\ &= \ e^{-r(t+\Delta t)} \, (1-p) \, \lambda''_t \, \kappa(t) \sigma \sqrt{\Delta t} \, \mathrm{Prob}\big(\{\pi_t(\omega) = m_\star\}\big) \\ &= \ \lambda_t \, \frac{K\sigma}{2e^r} \, \mathrm{Prob}\big(\{\pi_t(\omega) = m_\star\}\big) \, \sqrt{\Delta t} \end{aligned}$$

for some $\lambda_t \approx 1$. □

Corollary 7. *For every* $\tau \approx 0$ *we have* $\sum_{0 \leq t < \tau} \Upsilon(t) \ \approx \ 0$.

Proof. Without loss of generality, we assume that $\tau = n\Delta t \in \mathbb{S}$.

We use Proposition 4, 5 and 6:

$$\begin{aligned}\sum_{0\le t<\tau}\Upsilon(t) &= \sum_{t<\tau,\, t\in\mathbb{S}} \lambda_t\,\frac{K\sigma}{2e^r}\,\mathrm{Prob}\big(\{\pi_t(\omega)=m_\star\}\big)\,\sqrt{\Delta t}\\ &\le \sum_{t<\tau,\, t\in\mathbb{S}} \lambda_t\,\frac{K\sigma}{2e^r}\,\frac{2}{m_\star}\,\sqrt{\Delta t}\\ &= \frac{K\sigma}{e^r}\,\frac{1}{m_\star\sqrt{\Delta t}}\sum_{t<\tau,\, t\in\mathbb{S}}\lambda_t\,\Delta t,\end{aligned}$$

so the conclusion follows from Lemma 2 (II) (i) and $\lambda_t \approx 1$. □

We apply the following form of the Stirling's formula

$$k! = \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k e^{\epsilon/k} \quad \text{for some} \quad 0<\epsilon<\frac{1}{12}$$

and obtain for $0\le m\le n$ in ${}^*\mathbb{N}$ that

$$\begin{aligned}\binom{n}{\frac{n+m}{2}} &= \sqrt{\frac{2n}{\pi}}\,\frac{1}{\sqrt{n^2-m^2}}\,\frac{(2n)^n}{(n+m)^{\frac{n+m}{2}}\,(n-m)^{\frac{n-m}{2}}}\,\lambda'_{n,m}\\ &= \sqrt{\frac{2}{\pi n}}\Big(1-\Big(\frac{m}{n}\Big)^2\Big)^{-\frac12}\left(\frac{n^n}{(n+m)^{\frac{n+m}{2}}(n-m)^{\frac{n-m}{2}}}\right)2^n\,\lambda'_{n,m}\end{aligned}$$

for some $\lambda'_{n,m}$ which is ≈ 1 whenever both n and $n-m$ are infinite. For $t=n\Delta t\in\mathbb{S}$, let $\lambda_t = \lambda'_{n,m_\star}\Big(1-\Big(\frac{m_\star}{n}\Big)^2\Big)^{-\frac12}$. As a consequence of Proposition 2 (II)(ii) and the definition of $\mathbb{S}$, $\lambda_t\approx 1$ if $t\in\mathbb{S}$.

Therefore by Proposition 5 and 6 that when $t\in\mathbb{S}$

$$\Upsilon(t) = \lambda_t\,\frac{K\sigma}{\sqrt{2\pi}\,e^r}\,\frac{\Psi(t)}{\sqrt{t}}\,\Delta t, \qquad \text{for some} \quad \lambda_t\approx 1,$$

where for $t=n\Delta t\in\mathbb{S}$, we define

$$\Psi(t) := \left(\frac{n^n}{(n+m_\star)^{\frac{n+m_\star}{2}}(n-m_\star)^{\frac{n-m_\star}{2}}}\right)\Big(2^n\,p^{\frac n2}(1-p)^{\frac n2}\Big)\Big(\frac{p}{1-p}\Big)^{\frac{m_\star}{2}}.$$

Exponential form for $\Upsilon(t)$.

Lemma 8. *Let $0\not\approx t=n\Delta t\in\mathbb{S}$. Then*

(i) $\displaystyle \frac{n^n}{(n+m_\star)^{\frac{n+m_\star}{2}}(n-m_\star)^{\frac{n-m_\star}{2}}} \approx e^{-\frac{m_\star^2\Delta t}{2t}}$;

(ii) $2^n\,p^{\frac n2}(1-p)^{\frac n2} \approx e^{-\frac{\sigma^2 t}{8}}$;

(iii) $\displaystyle \Big(\frac{p}{1-p}\Big)^{\frac{m_\star}{2}} \approx e^{-\frac{\sigma}{2}m_\star\sqrt{\Delta t}}$.

Proof. (i): We have

$$\frac{n^n}{(n+m_\star)^{\frac{n+m_\star}{2}}(n-m_\star)^{\frac{n-m_\star}{2}}} \;=\; \Big(1-\Big(\frac{m_\star}{n}\Big)^2\Big)^{-\frac{n}{2}}\Big(\frac{n-m_\star}{n+m_\star}\Big)^{\frac{m_\star}{2}}.$$

By Proposition 2 (II)(i), $\dfrac{m_\star^2}{n} \;=\; \dfrac{m_\star^2\Delta t}{t}$ is finite, so

$$\Big(1-\Big(\frac{m_\star}{n}\Big)^2\Big)^{-\frac{n}{2}} \approx e^{\frac{m_\star^2}{2n}} = e^{\frac{m_\star^2\Delta t}{2t}}.$$

For the second term we have

$$\Big(\frac{n-m_\star}{n+m_\star}\Big)^{\frac{m_\star}{2}} \;=\; \Big(1-\frac{2m_\star}{n+m_\star}\Big)^{\frac{m_\star}{2}} \;=\; \Big(1-\frac{2m_\star\sqrt{\Delta t}}{\sqrt{nt}+m_\star\sqrt{\Delta t}}\Big)^{\frac{m_\star}{2}} \;\approx\; e^{-\frac{m_\star^2\Delta t}{t}},$$

so the result follows.

For (ii), we can perform a power series expansion and get

$$\ln\big(4p(1-p)\big) \;=\; \ln\left(4\Big(\frac{1}{1+e^{\sigma\sqrt{\Delta t}}}\Big)\Big(\frac{e^{\sigma\sqrt{\Delta t}}}{1+e^{\sigma\sqrt{\Delta t}}}\Big)\right) \;=\; -\frac{\sigma^2}{4}\Delta t \,+\, \epsilon\,\Delta t,$$

for some $\epsilon \,\approx\, 0$. That is

$$\frac{n}{2}\,\ln\big(4p(1-p)\big) \;\approx\; -\frac{\sigma^2\,t}{8}.$$

So the conclusion follows by raising both sides of the equation to the power of e.

(iii) follows from $\dfrac{p}{1-p} \;=\; e^{-\sigma\sqrt{\Delta t}}$. □

Therefore it follows from Lemma 8 that for $0 \not\approx t = n\Delta t \in \mathbb{S}$,

$$\begin{aligned}\Psi(t) \;&\approx\; \exp\Big(-\frac{t}{2}\Big(\frac{m_\star^2\Delta t}{t^2} \,+\, \frac{\sigma^2}{4} \,+\sigma\frac{m_\star\sqrt{\Delta t}}{t}\Big)\Big)\\ &=\; \exp\Big(-\frac{t}{2}\Big(\frac{m_\star\sqrt{\Delta t}}{t}+\frac{\sigma}{2}\Big)^2\Big)\\ &\approx\; \exp\Big(-\frac{t}{2}\Big(\frac{\ln K-r}{t\sigma}+\frac{\sigma}{2}\Big)^2\Big) \quad \text{(by Lemma 2 (II)).}\end{aligned}$$

Hyperfinite Riemann sum.

Now define the following standard positive function of t :

$$\Theta(t) \;=\; \frac{1}{\sqrt{t}}\,e^{-\frac{t}{2}\left(\frac{\ln K-r}{t\sigma}+\frac{\sigma}{2}\right)^2}.$$

Using the above, we have for all $n \in \mathbb{N}$ that

$$\frac{K\sigma}{\sqrt{2\pi}\,e^r}\sum_{n^{-1}<t\in\mathbb{S}}\frac{\Psi(t)}{\sqrt{t}}\,\Delta t \quad \text{is infinitely equivalent to} \quad \frac{K\sigma}{\sqrt{2\pi}\,e^r}\sum_{n^{-1}<t\in\mathbb{S}}\Theta(t)\,\Delta t,$$

consequently

$$\sum_{n^{-1}<t<1}\Upsilon(t) \quad \text{is infinitely equivalent to} \quad \frac{K\sigma}{\sqrt{2\pi}\,e^r}\sum_{n^{-1}<t\in\mathbb{S}}\Theta(t)\,\Delta t.$$

As a hyperfinite Riemann sum, we have

$$\frac{K\sigma}{\sqrt{2\pi}\, e^r} \sum_{n^{-1}<t\in\mathbb{S}} \Theta(t)\,\Delta t \approx \frac{K\sigma}{\sqrt{2\pi} e^r} \int_{n^{-1}}^{1} \frac{1}{\sqrt{t}}\, e^{-\frac{t}{2}\left(\frac{\ln K - r}{t\sigma} + \frac{\sigma}{2}\right)^2}\, dt \,<\, \infty,$$

and the integrals converges as $n \to \infty$ in $\mathbb{N}$. Now we can conclude from Corollary 7 and the overspill principle that (1) is infinitely equivalent to

$$\frac{K\sigma}{\sqrt{2\pi}\, e^r} \sum_{t\in\mathbb{S}} \Theta(t)\,\Delta t, \tag{2}$$

Finally we consider the sum according to the parity of $m_\star$. By Proposition 4, if $m_\star$ is even, then (2) is infinitely close to

$$\frac{K\sigma}{\sqrt{2\pi}\, e^r} \sum_{\text{even } t\in\mathbb{T}} \Theta(t)\,\Delta t \;=\; \frac{K\sigma}{2\sqrt{2\pi}\, e^r} \sum_{0\le t<1,\, t \text{ even}} \Theta(t)\,\Delta u, \tag{3}$$

where $\Delta u := 2\Delta t$.

Similarly, if $m_\star$, is odd (2) is infinitely close to

$$\frac{K\sigma}{\sqrt{2\pi}\, e^r} \sum_{\text{odd } t\in\mathbb{T}} \Theta(t)\,\Delta t \;=\; \frac{K\sigma}{2\sqrt{2\pi}\, e^r} \sum_{0\le t<1,\, t \text{ odd}} \Theta(t)\,\Delta u. \tag{4}$$

When viewed as a hyperfinite Riemann sum (which is finite), (3) and (4) are both infinitely close to the integral below and so the following is proved:

Theorem 9. *The expected cost of the SLSG strategy is infinitely close to the following integral*

$$\frac{K\sigma}{2\sqrt{2\pi} e^r} \int_0^1 \frac{1}{\sqrt{t}}\, e^{-\frac{t}{2}\left(\frac{\ln K - r}{t\sigma} + \frac{\sigma}{2}\right)^2}\, dt, \tag{5}$$

where $S_0 = 1$, $r \geq 0$ *is the riskless interest rate,* $\sigma > 0$ *the volatility of the stock and* $K > e^r$ *the critical level at terminal time* 1.

□

By changing the variable with $x = \sigma\sqrt{t}$, the above (5) equals

$$\frac{K}{\sqrt{2\pi} e^r} \int_0^{\sigma} e^{-\frac{1}{2}\left(\frac{\ln K - r}{x} + \frac{x}{2}\right)^2}\, dx. \tag{6}$$

Write the accumulated normal distribution function as

$$\mathcal{N}(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du.$$

We now show that (6) gives the Black-Scholes price.

Corollary 10. *The Stop-Loss-Start-Gain strategy has a cost given by the Black-Scholes formula.*

Proof. The classical formula discovered by Black and Scholes in [2] is

$$\mathcal{N}\Big(\frac{r-\ln K}{\sigma}+\frac{\sigma}{2}\Big)-Ke^{-r}\mathcal{N}\Big(\frac{r-\ln K}{\sigma}-\frac{\sigma}{2}\Big).$$

By differentiating it with respect to σ and then integrating back using integrating constant 0 (since when $\sigma=0$, the Black-Scholes price is 0), we get precisely (6). □

References

1. S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn and T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986.
2. F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 81, (1973), 637-659.
3. P.P. Carr, European Option Valuation when Carrying Costs are Unknown, unpublished notes, (1990).
4. P.P. Carr and R.A. Jarrow, The Stop-Loss Start-Gain Paradox: A New Decomposition into Intrinsic and Time Value, Rev. Financial Stud. 3(3), (1990), 469-492.
5. Cox, J.C., S.A. Ross, and M. Rubinstein, Option Pricing: A Simplified Approach, Journal of Financial Economics 7, (1979), 229-263.
6. N. Cutland, P.E. Kopp, W. Willinger, A Nonstandard Approach to Option Pricing, Mathematical Finance, 1(4) (1991), 1-38.
7. Y. Hu, B. Øksendal and D.M. Salopek, Weighted Local Time for Fractional Brownian Motion and Applications to Finance, University of Oslo, Pure Mathematics Preprint No. 13, May 2001.
8. A.E. Hurd & P.A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, New York, 1985.
9. S.-A. Ng, *Hypermodels in Mathematical Finance*, World Scientific, Singapore, 2003.
10. D.M. Salopek, A New Class of Nearly Self-financing Strategies, Statisctics & Probability Letters, 56 (2002), 69-75.
11. E. Seidenverg, A Case of Confused Identity, Financial Analysts Journal, (1988), 63-67.