# Some properties of nonstandard hulls of Banach algebras [1]

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Abstract

*We introduce the notion of generalized invertibility, a weak notion of invertibility that makes sense in the nonstandard hull of a Banach algebra. We show that a standard algebra is the closure of its invertible elements exactly when all the elements of its nonstandard hull are generalized invertible. For commutative algebras, we relate the problem of liftability of homomorphisms to the property of approximately multiplicative functionals being near to multiplicative functionals (AMNM property) and we prove a nonstandard characterization of the latter property. Finally we show that a standard superreflexive Banach algebra has the the AMNM property exactly when its nonstandard hull does.*

## 1. Introduction and preliminaries

Nonstandard methods (in the sense of A. Robinson) can be used to provide characterizations of standard properties. These characterizations usually refer to notions that are not available in the standard setting. In this paper we provide some examples of nonstandard characterizations in the framework of Banach algebras and their *nonstandard hulls*. For a detailed account of the nonstandard hull construction see [5] or [4] (such a construction will be outlined below for sake of fixing the notation). We recall that *ultraproducts* of Banach algebras are examples of nonstandard hulls. Therefore we may say that we study properties of ultraproducts of Banach algebras, a well known construction in functional analysis. Accordingly, the paper could have been written in ultraproduct language, but also, for the most part, in the language of *continuous logic* (see [2]). Our choice is motivated mainly by our familiarity with the nonstandard methods.

In §2 we relate density of the invertible elements in a Banach algebra to a weaker notion of invertibility, called *generalized invertibility,* that applies to the elements of its nonstandard hull. In §3 we study the *liftability* of homomorphisms in a nonstandard hull. We focus on commutative *AMNM* algebras (see [7]), for which we provide a nonstandard characterization. Finally we show that a standard superreflexive Banach algebra has the *AMNM* property exactly when its nonstandard hull does.

We assume familiarity with the basics of nonstandard analysis, a good reference can be found in [1]. Sufficient saturation of the nonstandard universe is used throughout.

We always work with Banach algebras over the complex field with unit element $e$. Throughout this paper, $X$ denotes an arbitrary internal or external Banach algebra and $\mathrm{Inv}(X)$ denotes the set of its invertible elements. We denote by $\overline{\mathrm{Inv}(X)}$ the closure of $\mathrm{Inv}(X)$. Given an internal normed algebra $X$, the *finite part* of $X$ is

$$\mathrm{Fin}(X) := \{x \in X : \ \|x\| < n \text{ for some } n \in \mathbb{N}\}.$$

The nonstandard hull $\widehat{X}$ is the set $\mathrm{Fin}(X)/\approx$, where $x \approx y$ means $\|x - y\| \approx 0$, together with operations induced by the equivalence relation and with norm given

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by the standard part of the $X$–norm. For simplicity, we denote both norms in the same way.

For $x \in X$, the equivalence class of $x$ is called the *monad* of $x$ and is denoted by $\widehat{x} := \{y \in X \mid y \approx x\}$. Saturation ensures that $\widehat{X}$ is a complete normed linear space, indeed a standard Banach algebra. The following holds:

**Theorem 1.** *Let $x \in \mathrm{Fin}(X)$, where $X$ is an internal normed algebra. Then*

$$\widehat{x} \in \mathrm{Inv}(\widehat{X}) \;\Leftrightarrow\; x \in \mathrm{Inv}(X) \text{ and } x^{-1} \in \mathrm{Fin}(X).$$

*Proof.* ($\Rightarrow$) Let $z \in \mathrm{Fin}(X)$ be such that $xz \approx e \approx zx$. First $xz \in \mathrm{Inv}(X)$ and $u := (xz)^{-1} = \sum_{n=0}^{\infty}(e-xz)^n$. Since $\|e-xz\| = \epsilon \approx 0$, we have $\|u\| \leq \sum_{n=0}^{\infty} \epsilon^n \approx 1$, so $u \in \mathrm{Fin}(X)$, $x(zu) = e$ and $zu \in \mathrm{Fin}(X)$. Secondly, $zx \in \mathrm{Inv}(X)$, so $v := (zx)^{-1} = \sum_{n=0}^{\infty}(e-zx)^n$. Since $\|e-zx\| \approx 0$ so $\|v\| \lessapprox 1$, hence $v \in \mathrm{Fin}(X)$, $(vz)x = e$ and $vz \in \mathrm{Fin}(X)$. Consequently $x$ is both left- and right-invertible and so $x^{-1} = zu = vz \in \mathrm{Fin}(X)$. □

**Corollary 2.** *Let $\widehat{x} \in \mathrm{Inv}(\widehat{X})$. Then $y \in \mathrm{Inv}(X)$ and $y^{-1} \in \mathrm{Fin}(X)$ for all $y \approx x$.*

If $X$ is a standard normed algebra, we write $\widehat{X}$ instead of $\widehat{{}^*X}$ for the canonical nonstandard hull extension.

We recall that a *homomorphism* $f$ of a Banach algebra $X$ is a multiplicative linear mapping, i.e. $f$ satisfies $f(xy) = f(x)f(y)$ for all $x, y \in X$. It is well known that any such $f$ is continuous. Moreover $f(e) = 1$ and $|f(x)| \leq \|x\|$ holds for all $x$. Therefore $\|f\| = 1$ for all homomorphisms $f$.

We denote by $\mathrm{hom}(X)$ the family of nonzero complex homomorphisms of $X$, by $0$ the zero homomorphism (the context will prevent any ambiguity), and we let $\underline{\mathrm{hom}}(X) = \mathrm{hom}(X) \cup \{0\}$.

If $Y$ is an internal Banach algebra and $f \in {}^*\mathrm{hom}(Y)$, the mapping $\widehat{f} : \widehat{Y} \to \mathbb{C}$ defined by $\widehat{f}(\widehat{y}) = {}^\circ f(y)$ is a well–defined complex homomorphism ( $^\circ$ denotes the *standard part* map).

## 2. Generalized invertibility

We define a generalization of the notion of invertibility in the nonstandard hull of a Banach algebra and we relate it to the density of invertible elements of the algebra.

**Definition 3.** *Let $X$ be an internal normed algebra. An element $\hat{x} \in \hat{X}$ is called generalized invertible if there exists $y \in \widehat{x} \cap \mathrm{Inv}(X)$. The set of generalized invertible elements in $\hat{X}$ is denoted by* $\mathrm{GInv}(\hat{X})$.

It follows from Theorem 1 that $\mathrm{Inv}(\widehat{X}) \subseteq \mathrm{GInv}(\widehat{X})$. Furthermore the inclusion is strict, since $0 \in \mathrm{GInv}(\hat{X})$. Note also that all elements in $\mathrm{GInv}(\widehat{X}) \setminus \mathrm{Inv}(\widehat{X})$ are *algebraic* (equivalently: *topological*) divisors of zero. For, let $\widehat{x} \in \mathrm{GInv}(\widehat{X}) \setminus \mathrm{Inv}(\widehat{X})$ and let $x \approx y \in \mathrm{Inv}(X)$. Note that $y^{-1}$ has infinite norm. Let $z = \|y^{-1}\|^{-1} y^{-1}$. Then $xz = (x-y)z + yz \approx 0$. Similarly, $zx \approx 0$.

In some specific cases, the existence of "nontrivial" generalized invertible elements can be easily proved. For instance, all elements in $\widehat{\ell_\infty(\mathbb{N})}$ are generalized invertible.

**Proposition 4.** *Let $X$ be a standard Banach algebra. Identifying $x \in X$ with its image $\widehat{x}$ in $\widehat{X}$, we have $x \in \overline{\mathrm{Inv}(X)}$ if and only if $x \in \mathrm{GInv}(\widehat{X})$, namely $\overline{\mathrm{Inv}(X)} = X \cap \mathrm{GInv}(\widehat{X})$.*

*Proof.* ($\Rightarrow$) Let $y_n \in \mathrm{Inv}(X)$ and $\|x-y_n\| \to 0$. By saturation, there is $y \in \mathrm{Inv}({}^*X)$ such that $x \approx y$, hence $x = \widehat{y} \in \mathrm{GInv}(X)$.
($\Leftarrow$) Let $y \in \mathrm{Inv}({}^*X)$ be such that $x \approx y$. By Transfer, for all $n \in \mathbb{N}^+$ there exists $y_n \in \mathrm{Inv}(X)$ such that $\|x - y_n\| \leq 1/n$. □

The next proposition yields at once a nonstandard characterization of density of the group of invertible elements in a standard Banach algebra $Y$: $\mathrm{Inv}(Y)$ is dense in $Y$ precisely when all the elements of the nonstandard hull $\widehat{Y}$ are generalized invertible. Consequently, there are non generalized invertible elements in the nonstandard hull of any standard Banach algebra $Y$ such that $\overline{\mathrm{Inv}(Y)} \neq Y$.

**Proposition 5.** *Let $\mathcal{S}_Y$ be the unit sphere of a standard Banach algebra $Y$ and let $X = \widehat{Y}$. Then $X = \mathrm{GInv}(X)$ if and only if $\mathcal{S}_Y \subseteq \overline{\mathrm{Inv}(Y)}$.*

*Proof.* ($\Rightarrow$) Assume $\mathcal{S}_Y \not\subseteq \overline{\mathrm{Inv}(Y)}$. Then, for some $x \in \mathcal{S}_Y$ and some standard $r > 0$, we have $\|x - y\| \geq r$ for all $y \in \overline{\mathrm{Inv}(Y)}$. Applying Transfer, we get $\|x - y\| \geq r$ for all $y \in \overline{\mathrm{Inv}({}^*Y)}$. Since $r \not\approx 0$, it follows that $\widehat{x} \notin \mathrm{GInv}(X)$.
($\Leftarrow$) By Transfer, ${}^*\mathcal{S}_Y$ is contained into the internal closure of $\mathrm{Inv}({}^*Y)$ so for all $x \in {}^*\mathcal{S}_Y$ there is $y \in \mathrm{Inv}({}^*Y)$ such that $x \approx y$. (Recall also that $0 \in \mathrm{GInv}(X)$.) □

By saturation, the set $\mathrm{GInv}(\widehat{X})$ is a closed multiplicative monoid in every internal normed algebra $X$ (recall that the group of invertible elements in a Banach algebra is open). It follows that $\overline{\mathrm{Inv}(\widehat{X})} \subseteq \mathrm{GInv}(\widehat{X})$, but the converse inclusion does not hold in general: a counterexample can be found in $\widehat{\ell_1(N)}$, with $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and product given by convolution.

Let $X$ be a standard Banach algebra. From Proposition 5, the closedness of $\mathrm{GInv}(\widehat{X})$ and from $\overline{\mathrm{Inv}(\widehat{X})} \subseteq \mathrm{GInv}(\widehat{X})$, we get that $\overline{\mathrm{Inv}(X)} = X$ whenever $\overline{\mathrm{Inv}(\widehat{X})} = \widehat{X}$.

**Question 6.** *Suppose $X$ is a standard Banach algebra such that $\overline{\mathrm{Inv}(X)} = X$. Does it follow that $\overline{\mathrm{Inv}(\widehat{X})} = \widehat{X}$?*

Note that the condition $\overline{\mathrm{Inv}(Y)} = Y$ does not imply $\overline{\mathrm{Inv}(\widehat{Y})} = \widehat{Y}$ for all internal Banach algebras $Y$. A counterexample is given by $\ell_1(N)$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$: one hand, the property of density of the group of invertible elements of $\ell_1(n)$, for all $n \in \mathbb{N}$, transfers to $\ell_1(\mathbb{N})$; on the other hand we have previously remarked that $\overline{\mathrm{Inv}(\widehat{X})} \subsetneq \mathrm{GInv}(\widehat{X})$.

## 3. Liftable homomorphisms and the *AMNM* property

*In this section all Banach algebras are assumed to be commutative.*

We study the liftability of homomorphisms in a nonstandard hull and the *AMNM* Banach algebras, where almost multiplicative mappings are near multiplicative.

It is well known that an element $x$ of a standard *commutative* Banach algebra $X$ is invertible if and only if $f(x) \neq 0$ for all $f \in \hom(X)$. We want to investigate whether a similar characterization holds for generalized invertible elements.

The following notion is crucial for the application of nonstandard techniques:

**Definition 7.** *Let $X$ be an an internal Banach algebra. We say that*

(1) *$f \in (\widehat{X})'$ is $l$–liftable if there exists $g \in X'$ (the internal set of $*$-linear bounded functionals of $X$) such that $f(\widehat{x}) \approx g(x)$ for all $x \in \mathrm{Fin}(X)$. We say that $g$ as above is an $l$–lifting of $f$.*

(2) *$f \in \underline{\mathrm{hom}}(\widehat{X})$ is $h$–liftable if there exists $g \in \underline{\mathrm{hom}}(X)$ (the internal set of $*$–complex homomorphisms of $X$) such that $f(\widehat{x}) \approx g(x)$ for all $x \in \mathrm{Fin}(X)$. We say that $g$ as above is an $h$–lifting of $f$.*

*In both cases we write $f = \widehat{g}$.*

**Definition 8.** *Let $X$ be an internal Banach algebra. We say that $g \in X'$ is almost multiplicative (briefly: a.m.) if $g(xy) \approx g(x)g(y)$ for all $x, y \in \mathrm{Fin}(X)$.*

We recall a definition from [7]: a linear bounded functional on a standard Banach algebra $X$ is called *$\epsilon$–multiplicative*, where $\epsilon$ is a positive real number, if

$$|f(xy) - f(x)f(y)| \le \epsilon \|x\| \|y\| \quad \text{for all } x, y \in X.$$

We denote by $\mathrm{hom}_\epsilon(X)$ the set of $\epsilon$–multiplicative functionals.

Note that the $l$–lifting of a homomorphism is *a.m.* Moreover the *a.m.* internal functionals are exactly the $\epsilon$–multiplicative ones, for some infinitesimal $\epsilon$. As for the nontrivial inclusion, let $g$ be an *a.m.* functional on an internal Banach algebra $X$. Let $\epsilon$ be the internal supremum of the set $\{\, |g(xy) - g(x)g(y)| : \ x, y \in X \text{ and } \|x\| = \|y\| = 1 \,\}$. Then $\epsilon \approx 0$ and $|g(xy) - g(x)g(y)| \le \epsilon \|x\| \|y\|$, for all $x, y \in X$.

Furthermore, suppose $g$ is *a.m.* By the previous remark and by Transfer of Proposition 5.5 in [6], we get $\|g\| \le 1 + \epsilon$ for some infinitesimal $\epsilon$. Moreover if $g(e) \not\approx 0$ then $g(e) \approx 1$ and so $\|g\| \approx 1$. Note also that $g(e) \approx 0$ is equivalent to $g(x) \approx 0$ for all $x \in \mathrm{Fin}(X)$.

Invertibility within the monad of a point and liftable homomorphisms are related:

**Proposition 9.** *Let $0 \neq \widehat{x} \in \widehat{X}$, where $X$ is an internal Banach algebra. Then there exists noninvertible $y \in \widehat{x}$ if and only if $f(\widehat{x}) = 0$ for some $h$–liftable $f \in \mathrm{hom}(\widehat{X})$.*

*Proof.* ($\Leftarrow$) Let $g$ be an $h$–lifting of $f \in \mathrm{hom}(\widehat{X})$ such that $f(\widehat{x}) = 0$. Then $g(x) = \epsilon$ for some $\epsilon \approx 0$. The element $y = (1 - \epsilon)x$ is in $\widehat{x}$ and $g(y) = 0$. Hence $y \notin \mathrm{Inv}(X)$. □

**Corollary 10.** *Let $0 \neq \widehat{x}$ be an element of the nonstandard hull $\widehat{X}$ of an internal Banach algebra $X$. If $f(\widehat{x}) \neq 0$ for all $h$–liftable $f \in \mathrm{hom}(\widehat{X})$ then $\widehat{x} \in \mathrm{GInv}(\widehat{X})$.*

The converse implication in Corollary 10 does not hold in general: let $X = {}^*\ell_\infty(\mathbb{N})$ and let $f : (a_n)_{n \in {}^*\mathbb{N}} \mapsto a_0$. Clearly $\widehat{f}$ is $h$–liftable. Let $x = (0, 1, 1, \dots)$. Then $0 \neq \widehat{x} \in \mathrm{GInv}(\widehat{X})$ and $\widehat{f}(\widehat{x}) = 0$.

We point out that liftability of linear functionals is an important property of a nonstandard hull: the nonstandard hull $\widehat{X}$ of an internal Banach space $X$ is reflexive (equivalently: superreflexive) if and only all its bounded linear functionals are $l$–liftable, namely if and only if $(\widehat{X})' = \widehat{(X')}$ (see, for instance, [5]).

Next we provide a necessary condition for the $l$–liftability of all homomorphisms that refers to the *joint spectrum*. We recall that the joint spectrum $\sigma(\bar{a})$ of a tuple $\bar{a} = (a_1, \dots, a_n) \in X^n$ is the set $\{(f(a_1), \dots, f(a_n)) : f \in \mathrm{hom}(X)\}$.

**Proposition 11.** *Let $X$ be an internal Banach algebra such that $\mathrm{hom}(\widehat{X}) \subseteq \widehat{(X')}$. Then $\sigma(\widehat{a_1}, \dots, \widehat{a_n}) = {}^\circ\{(g(a_1), \dots, g(a_n)) : g \in X', \|g\| = 1$ and $g$ is a.m.$\}$, for all $n \in \mathbb{N}$ and all $(a_1, \dots, a_n) \in \mathrm{Fin}(X)^n$.*

*Proof.* Let $\bar\lambda \in \sigma(\widehat{a_1}, \ldots, \widehat{a_n})$. Then there exists $f \in \hom(\widehat{X})$ such that $f(\widehat{a_i}) = \lambda_i$, $i = 1, \ldots, n$. Let $g$ be an $l$–lifting of $f$. Since $g$ is *a.m.*, we can assume without loss of generality that $\|g\| = 1$. Then $g(a_i) \approx \lambda_i$, $i = 1, \ldots, n$ and $\bar\lambda \in {}^\circ\{(g(a_1), \ldots, g(a_n)) : g \in X', \|g\| = 1 \text{ and } g \text{ is } a.m.\}$. Conversely, let $\bar\lambda \in {}^\circ\{(g(a_1), \ldots, g(a_n)) : g \in X', \|g\| = 1 \text{ and } g \text{ is } a.m.\}$ and let $g \in X'$ be an *a.m.* norm one functional such that $\lambda_i \approx g(a_i)$, $i = 1, \ldots, n$. Then $\widehat{g} \in \hom(\widehat{X})$ and $\widehat{g}(\widehat{a_i}) = \lambda_i$, $i = 1, \ldots, n$. Hence $\bar\lambda \in \sigma(\widehat{a_1}, \ldots, \widehat{a_n})$. □

We do not know whether the condition in the previous proposition suffices for the $l$–liftability of all homomorphisms. The condition $\sigma(\widehat{a}) = {}^\circ\{g(a) : g \in X', \|g\| = 1 \text{ and } g \text{ is } a.m.\}$ for all $a \in \mathrm{Fin}(X)$ does not, as shown by letting $X := {}^*\mathcal{C}(K)$, where $K$ is compact and Hausdorff. As a Banach algebra, $\widehat{X}$ is isometrically isomorphic to $\mathcal{C}(\widehat{K})$, for some compact Hausdorff space $\widehat{K}$ that contains ${}^*K$ as dense subset (see [4, §3]). Let $f \in X$. Then $\sigma(f) = \{f(k) : k \in {}^*K\}$ and $\sigma(\widehat{f}) = \{\widehat{f}(k) : k \in \widehat{K}\}$.

We claim that $\sigma(\widehat{f}) = {}^\circ\sigma(f) = {}^\circ\{g(f) : g \in X', \|g\| = 1 \text{ and } g \text{ is } a.m.\}$, for all $f \in X$. As for the inclusion $\sigma(\widehat{f}) \subseteq {}^\circ\sigma(f)$, let $k \in \widehat{K} \setminus {}^*K$. From the properties of $\widehat{f}$ and the density of ${}^*K$ in $\widehat{K}$, we get $\widehat{f}(\widehat{K}) = \overline{\widehat{f}({}^*K)}$. Let $(k_n)_{n\in\mathbb{N}} \subset {}^*K$ be such that $(\widehat{f}(k_n))_{n\in\mathbb{N}}$ converges to $\widehat{f}(k)$. By saturation there exists $h \in {}^*K$ such that $f(h) \approx \widehat{f}(k)$. Hence $\widehat{f}(k) \in {}^\circ\sigma(f)$. Moreover ${}^\circ\sigma(f) \subseteq {}^\circ\{g(f) : g \in X', \|g\| = 1 \text{ and } g \text{ is } a.m.\} \subseteq \sigma(\widehat{f})$ always hold. On the other hand, from Example 18 below we get that $\hom(\widehat{X}) \not\subseteq \widehat{(X')}$.

We recall another definition from [7]: a Banach algebra $X$ has the *AMNM property* (briefly: $X$ is *AMNM*) if

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall f \in \hom_\delta(X)\, \exists g \in \underline{\hom}(X)\, (\|f - g\| < \epsilon).$$

The following is a nonstandard characterization of the *AMNM* property:

**Theorem 12.** *Let $X$ be a standard Banach algebra. Then $X$ is AMNM if and only if* $\widehat{{}^*\underline{\hom}(X)} = \underline{\hom}(\widehat{X}) \cap \widehat{(X')}$.

*Proof.* (⇒) The inclusion $\widehat{{}^*\underline{\hom}(X)} \subseteq \underline{\hom}(\widehat{X}) \cap \widehat{(X')}$ always holds. As for the converse inclusion, let $f \in \underline{\hom}(\widehat{X}) \cap \widehat{(X')}$. Then there exists an $l$–lifting $g \in {}^*(X')$ of $f$, which is $\epsilon$–multiplicative for some $\epsilon \approx 0$. By Transfer of the property that $X$ is *AMNM*, we have $\mathrm{dist}(g, {}^*\underline{\hom}(X)) < 1/n$ for all $n \in \mathbb{N}^+$. By saturation there exists $h \in {}^*\underline{\hom}(X)$ such that $\|h - g\| \approx 0$. Hence $\widehat{h} = f$ and so $f \in \widehat{{}^*\underline{\hom}(X)}$.
(⇐) Suppose $X$ is not *AMNM* and let $\epsilon \in \mathbb{R}^+$ be such that for all $n \in \mathbb{N}^+$ there exists a norm one $1/n$–multiplicative linear functional $f_n$ such that $\mathrm{dist}(f_n, {}^*\underline{\hom}(X)) > \epsilon$. By saturation, there exist $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and a norm one $1/N$–multiplicative $f \in {}^*(X')$ such that $\mathrm{dist}(f, {}^*\underline{\hom}(X)) > \epsilon$. Hence $\widehat{f} \notin \widehat{{}^*\underline{\hom}(X)}$, but $\widehat{f} \in \underline{\hom}(\widehat{X}) \cap \widehat{(X')}$. □

**Corollary 13.** *Let $X$ be a standard AMNM Banach algebra and let $f \in \underline{\hom}(\widehat{X})$. Then $f$ is $h$–liftable if and only if it is $l$–liftable.*

*Proof.* Suppose $f$ is $l$–liftable. Then $f \in \underline{\hom}(\widehat{X}) \cap \widehat{(X')}$ and Theorem 12 applies. □

Next we investigate whether the *AMNM* property is preserved from nonstandard hull to algebra and conversely.

**Proposition 14.** *Let $X$ be a standard Banach algebra. If $\widehat{X}$ is AMNM then $X$ is AMNM.*

*Proof.* Let $\delta \in \mathbb{R}^+$ corresponding to $\epsilon \in \mathbb{R}^+$ in the definition of *AMNM* for $\widehat{X}$. By Transfer, each $f \in \underline{\hom}_\delta(X)$ extends to $\widehat{f} \in \underline{\hom}_\delta(\widehat{X})$ (regarding $X$ as a subalgebra of $\widehat{X}$). Let $g \in \underline{\hom}(\widehat{X})$ be such that $\|\widehat{f} - g\| < \epsilon$ and let $g' = g|_A$. Clearly $g' \in \underline{\hom}(X)$ and $\|f - g'\| < \epsilon$. □

**Proposition 15.** *Let $X$ be a standard AMNM Banach algebra. Suppose that for all $\delta \in \mathbb{R}^+$ all $\delta$–homomorphism of $\widehat{X}$ are $l$–liftable. Then $\widehat{X}$ is AMNM.*

*Proof.* Let $\epsilon \in \mathbb{R}^+$ and let $\delta' = \frac{1}{2}\delta(\epsilon/2)$, where $\delta(\epsilon/2)$ is a positive real number corresponding to $\epsilon/2$ in the definition of *AMNM* for $X$. Given $h \in \underline{\hom}_{\delta'}(\widehat{X})$, let $k \in {}^*(X')$ be an $l$–lifting of $h$. By easy calculation we get that $k \in {}^*\underline{\hom}_{\delta(\epsilon/2)}$. Hence, by Transfer of the *AMNM* property for $X$, there exists $g \in {}^*\underline{\hom}(X)$ such that $\|g - k\| < \epsilon/2$. The map $\widehat{g} : \widehat{X} \to \mathbb{C}$ defined by $\widehat{x} \mapsto {}^\circ g(x)$, $x \in \mathrm{Fin}({}^*X)$, is thus a homomorphism of $\widehat{X}$ and $\|h - \widehat{g}\| < \epsilon$. □

**Corollary 16.** *Let $X$ be a standard superreflexive AMNM Banach algebra. Then $\widehat{X}$ is AMNM.*

*Proof.* By [5, Corollary 3.9], $\widehat{X}$ is superreflexive. Then [5, Proposition 3.11] applies and all the assumptions of Proposition 15 are satisfied. □

**Corollary 17.** *Let $X$ be a superreflexive Banach algebra. Then $X$ is AMNM if and only if $\widehat{X}$ is AMNM.*

We finish with two examples of Banach algebras that behave differently with respect to $h$–liftability.

**Example 18.** We characterize the $h$–liftable homomorphisms of $\widehat{\mathcal{C}(K)}$, where $K$ is a compact Hausdorff space. The algebra $X := \mathcal{C}(K)$ is *AMNM* (see [7]). Moreover, $\widehat{X}$ is isometrically isomorphic to $\mathcal{C}(\widehat{K})$ for some compact Hausdorff space $\widehat{K}$ that contains ${}^*K$ as dense subset (see [4, §3]). For $k \in K$, let $\theta_k$ be the evaluation map at point $k$. The map $k \mapsto \theta_k$ is a bijection between $K$ and $\underline{\hom}(X)$. Same with ${}^*X$ and $\widehat{X}$.

Suppose $\theta_k \in \underline{\hom}(\widehat{X})$ is $h$–liftable and let $h \in {}^*K$ be such that $\theta_k(\widehat{f}) \approx \theta_h(f)$ for all $f \in \mathrm{Fin}({}^*X)$. Then $\widehat{f}(k) = \theta_k(\widehat{f}) = \widehat{\theta_h}(\widehat{f}) = {}^\circ\theta_h(f) = {}^\circ f(h) = \widehat{f}(h)$, for all $\widehat{f} \in \widehat{X}$, and so $k = h$. Therefore the $h$–liftable homomorphisms in $\underline{\hom}(\widehat{X})$ are exactly the evaluation maps at points of ${}^*K$. Furthermore, Corollary 13 implies that the evaluation maps at points of $\widehat{K} \setminus {}^*K$ are not even $l$–liftable. Note that the existence of non–$l$–liftable linear functionals in $\widehat{X}'$ follows from [5, Corollary 3.9, Proposition 3.1].

**Example 19.** We describe a nonstandard hull all of whose homomorphisms are $h$–liftable. We embed the Banach algebra $\ell_2 = \ell_2(\mathbb{N})$ (with pointwise multiplication) into a unit algebra in the usual way: as for the Banach space structure we let $X = \ell_2 \oplus_1 \mathbb{C}$ (the $\ell_1$–sum of $\ell_2$ and $\mathbb{C}$), with norm and multiplication given by $\|(x, a)\| = \|x\| + |a|$, and $(x, a) \cdot (y, b) = (xy + bx + ay, ab)$ respectively. The algebra $X$ is *AMNM* by [7, Proposition 4.2, Corollary 3.5]. Since $\widehat{X} = \widehat{\ell_2} \oplus_1 \mathbb{C}$ and the two summands are reflexive ([5, Corollary 3.9]), $\widehat{X}$ is reflexive. By [5, Proposition 3.11], the elements of $\underline{\hom}(\widehat{X})$ are $l$–liftable. Corollary 13 implies that they are $h$–liftable.

## References


1. S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn & T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986.
2. I. Ben Yaacov, A. Berenstein, C.W. Henson and A. Usvyatsov, Model theory for metric structures. In *Model theory with applications to algebra and analysis* vol. 2, 315–427, London Math. Soc. Lecture Note Ser., 350, Cambridge Univ. Press, Cambridge, 2008.
3. T.W. Dawson & J.S. Feinstein, On the denseness of the invertible group in Banach algebras, Proc. Amer. Math. Soc. 131–9 (2003), 2831–2839 (electronic).
4. C.W. Henson, Nonstandard hulls of Banach spaces, Israel J. Math. 25 (1976), 108–144.
5. C.W. Henson and L.C. Moore, Jr., Nonstandard analysis and the theory of Banach spaces, in Nonstandard analysis–recent developments (A.E. Hurd ed.), LNM 983 (1983), 27–112.
6. K. Jarosz, *Perturbations of Banach algebras*, LNM 1120, Springer-Verlag, Berlin, 1985.
7. B.E. Johnson, Approximately multiplicative functionals, J. Lond. Math. Soc. 34 (1986), 489–510.



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