Acta Mathematica Sinica, English Series
Feb., 2008, Vol. 24, No. 2, pp. 241–252

DOI: 10.1007/s10114-007-1024-7
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A Nonstandard Lévy–Khintchine Formula and Lévy Processes

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Abstract We use methods from nonstandard analysis to obtain a short and simple derivation of the Lévy–Khintchine formula via an explicit construction of certain laws of the infinitesimal increments. Consequently, any arbitrary Lévy process is representable as the standard part of a hyperfinite sum of infinitesimal increments.

Keywords Lévy processes, Lévy–Khintchine formula, nonstandard analysis, infinitesimal, hyperfinite

MR(2000) Subject Classification 03H05, 60G51; 60G50, 60E07, 47H05

1 Introduction

The main result of this paper is a nonstandard analysis version of the *Lévy–Khintchine formula* and the main application is a representation of an arbitrary Lévy process as the standard part of an internal process given by a hyperfinite sum of increments. The latter was achieved by Lindstrøm using a *hyperfinite discrete random walk* in [1]. There is also a related result by Di Nunno, Øksendal and Proske in [2] using the white noise approach. The current work was inspired by these predecessors but the approach differs from theirs by the adoption of a hyperfinite-dimensional Euclidean space ${}^*\mathbb{R}^{\mathbb{T}}$ for the increment sequences, resulting in a (non-discrete) *hyperreal random walk*. Although the space is not hyperfinite, we nevertheless deal with a hyperfinite sum of hyperreal increments. When the increments obey the same Gaussian law, the result is a Brownian motion, already produced by Cutland in [3] by modifying Anderson's Brownian motion.

In contrast to other approaches, we rely heavily on a nonstandard analysis version of the Lévy–Khintchine formula. A simple proof is given by using ${}^*\mathbb{R}^{\mathbb{T}}$ and an explicit construction of the law of the infinitesimal increments. We will show from a nonstandard point of view the amazing wealth of information codified in the Lévy–Khintchine formula: not only does it characterize the law of the standard part of the hyperfinite sum, it also ensures finiteness, stochastic continuity (in fact càdlàg) and determines the martingale part.

The advantage of using hyperreal random walk lies in the ready transferability of elementary calculus to the hyperfinite dimensional Euclidean space. This advantage should be even more prominent for the nonstandard development of the Malliavin Calculus (see [4–5]; cf. [1] for a discrete approach) and the White Noise Analysis, where it will be shown elsewhere that the Lévy–Khintchine formula is once again an important tool (cf. [6] for a sketch for the Gaussian

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Received September 8, 2005, Revised February 21, 2006, Accepted June 5, 2006
Work completed at BiBoS Universität Bielefeld in 2004 with support from Norway-SA Grant 2067063, acknowledged with gratitude

case and [2] for a standard approach). Another possible application is to use the hyperfinite-dimensional Euclidean space to investigate questions concerning the Hausdorff dimension of the inverse image of a set under a Lévy process as studied in [7].

Standard treatment of the Lévy–Khintchine formula can be found in [8]. An alternative non-probabilistic proof is given by [9].

Section 2 is a summary of the needed background from nonstandard analysis. Section 3 contains a proof of the nonstandard analysis version of the Lévy–Khintchine formula in the form of an infinitesimal generator. Section 4 uses hyperreal random walks to represent in law an arbitrary Lévy process by taking the standard part, and Section 5 uses nonstandard martingale theory to show that in fact the standard part is càdlàg, hence a Lévy process.

It is tempting to move further and study Lévy stochastic integrations, but this will be better done in a Malliavin setting elsewhere.

2 Preliminaries

This section is a brief review of some needed notions from nonstandard analysis.

The nonstandard extension of a standard mathematical object X is denoted by ${}^*X$. An element from some ${}^*X$ is referred to as an *internal* object. We identify a property with its underlying defining set, so we may speak of *properties such as *continuity and *differentiability. The extension from X to ${}^*X$ preserves first-order properties in the language of set theory and is done simultaneously for all mathematical objects under consideration. In particular we have new objects like ${}^*\mathbb{N}$ (hypernatural numbers) and ${}^*\mathbb{R}$ (hyperreal numbers), which still behave with respect to each other in the same formal manner as $\mathbb{N}$ and $\mathbb{R}$. Elements in the set ${}^*\mathbb{N}$ are called *hyperfinite*; a set counted internally by a hyperfinite number is also called hyperfinite (this is the same as *finite); given $r, s \in {}^*\mathbb{R}$, if $|r-s| < q$ for all $q \in \mathbb{R}^+$ (equivalently $|r-s| < 1/n$ for all $n \in \mathbb{N}^+$), we write $r \approx s$ (*infinitely close*); r is called *infinitesimal* when $r \approx 0$; a finite element r of ${}^*\mathbb{R}$ (written $|r| < \infty$) is one with $|r| < n$ for some $n \in \mathbb{N}$; such r is $\approx s$ for a unique $s \in \mathbb{R}$ (called the *standard part*; in symbol: $s = {}^\circ r$); write $r \approx \pm\infty$ when r is infinite. For convenience we set ${}^\circ r = \pm\infty$ for infinite r. We use similar notions for elements in $\mathbb{C}$ or ${}^*\mathbb{R}^n$.

We arbitrarily fix some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, an infinite hyperfinite number, and denote the infinitesimal time increment by $\Delta t := 1/N$. Then we form the hyperfinite timeline:

$$\mathbb{T} := \{0, \Delta t, 2\Delta t, \ldots, N\}.$$

The set $\mathbb{T}$ will be used as our nonstandard counterpart of the timeline $[0,\infty)$. Note that the internal cardinality of $\mathbb{T}$ is hyperfinite, in fact N^2+1.

We will assume throughout that the so-called *ω_1-saturation principle* is satisfied in our construction of nonstandard objects (which is possible under a weakened form of the Axiom of Choice), namely:

> *If $\mathscr{F}$ is a countable family of internal sets such that $\bigcap \mathscr{F}_0 \neq \emptyset$ for any finite subfamily $\mathscr{F}_0$ of $\mathscr{F}$, then $\bigcap \mathscr{F} \neq \emptyset$.*

An internal function $F : {}^*\mathbb{R} \to {}^*\mathbb{C}$ is called S-bounded if $|F(r)|$ is finite for all $r \in {}^*\mathbb{R}$. (Using ω_1-saturation this is equivalent to having some fixed finite bound for all $|F(r)|$.) It is called S-continuous if $r \approx s \Rightarrow F(r) \approx F(s)$ for all $r, s \in {}^*\mathbb{R}$. Note the difference between *continuity and S-continuity. If F is S-continuous, it can be proved that there is a unique continuous $f : \mathbb{R} \to \mathbb{C}$ such that $f(u) = {}^\circ(F(t))$ whenever $t \approx u$. In this case we write $f = {}^\circ F$. The notion of S-boundedness and S-continuity extends naturally to functions like $F : \mathbb{T} \to {}^*\mathbb{R}^m$ or $F : {}^*\mathbb{R}^n \to {}^*\mathbb{R}^m$.

Given an internal probability measure μ on an internal space Ω equipped with algebra $\mathscr{B}$ (so $\mu : \mathscr{B} \to {}^*[0,1]$ is $^*\sigma$-additive), we obtain a finitely additive probability ${}^\circ\mu$ by taking the standard part of measure values. As a consequence of the ω_1-saturation and the Carathéodory Theorem, one can prove that ${}^\circ\mu$ has a unique extension which is a σ-additive probability, denoted by μ_L, called the *Loeb measure* of μ after its inventor P. Loeb.

The nonstandard counterpart of integration theory is the following. An internal function $F : \Omega \to \mathbb{R}^n$ for some $n \in \mathbb{N}$ is called S-integrable if $\mathbb{E}\left[|F|\right] < \infty$ and $\mathbb{E}_C\left[|F|\right] \approx 0$ whenever $\mu(C) \approx 0$. We say that F is SL^p for some $p \in {}^*\mathbb{R}^+$, if $|F|^p$ is S-integrable. A useful result is the following lemma due to T. Lindstrøm:

If F is SL^p for some ${}^\circ p > 1$, then F is S-integrable.

The F is said to be a *lifting* of some standard $f : \Omega \to \mathbb{R}^n$, if $F(\omega) \approx f(\omega)$ *a.s.* with respect to the Loeb measure.

We use the same definitions for other internal functions such as $F : \Omega \to \mathbb{C}^n$.

Further introduction to nonstandard analysis can be found in [10–11] and [6].

Our terminology from Lévy processes follows that of [8] and [12]. In particular, by a *Lévy process* we mean a time-homogeneous stochastic process starting from 0 with independent increments which is a.s. right-continuous with left-limit (càdlàg), and it is a *Lévy process in law* if càdlàg is weakened to stochastic continuity; see [13] (especially the introductory article by Sato), [15] and [14].

3 Lévy–Khintchine Formula

In this section we produce the main tool of the paper, Theorem 1, namely, the nonstandard analysis version of the Lévy–Khintchine formula in operator form.

We use the same fixed $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $\Delta t = 1/N$ and $\mathbb{T}$ as before.

Given an internal probability measure δ on ${}^*\mathbb{R}^d$, $d \in \mathbb{N}^+$, we define the following operator on ${}^*\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$, the space of internal functions from ${}^*\mathbb{R}^d$ into ${}^*\mathbb{C}$ with bounded and continuous derivatives of order ≤ 2 :

$$L\phi(x) = \int_{{}^*\mathbb{R}^d} \frac{\phi(x+y) - \phi(x)}{\Delta t}\, \delta(dy), \qquad \phi \in {}^*\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C}).$$

We call L the *generator* associated with δ. (Very often we drop the $*$ from ${}^*\phi$ when there is no danger of confusion.)

By a Lévy triple we mean some $(\gamma, \mathbb{A}, \nu)$, where $\gamma \in \mathbb{R}^d$, $\mathbb{A}$ is a standard $d \times d$ nonnegative definite symmetric matrix and ν is a standard Borel measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \frac{|y|^2}{1+|y|^2}\nu(dy) < \infty$ (cf. [8] Remark 8.4 for other equivalent conditions). Note that the above implies, for all $\phi \in {}^*\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$ and fixed $x \in {}^*\mathbb{R}^d$, the function $y \mapsto \big(\phi(x+y) - \phi(x) - \frac{\langle y, \nabla\phi(x)\rangle}{1+|y|^2}\big)$ is ${}^*\nu$-integrable, with finite integral over ${}^*\mathbb{R}^d$ if in addition ϕ has S-bounded derivatives of order 0, 1 and 2. Moreover, for $\phi \in \mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$ and $x \in \mathbb{R}^d$, $y \mapsto {}^\circ\big({}^*\phi(x+y) - {}^*\phi(x) - \frac{\langle y, \nabla\, {}^*\phi(x)\rangle}{1+|y|^2}\big)$ is ν-integrable.[1]

The following is our nonstandard version of the Lévy–Khintchine formula in operator form:

Theorem 1 *Let* $(\gamma, \mathbb{A}, \nu)$ *be a standard Lévy triple. Write* $\mathbb{A} = [a_{i,j}]_{1\le i,j\le d}$. *Then there is an internal probability measure* δ *on* ${}^*\mathbb{R}^d$ *so that the associated generator* L *satisfies, for each* $\phi \in {}^*\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$ *with S-bounded and S-continuous derivatives of order* 0, 1 *and* 2, *and* $x \in {}^*\mathbb{R}^d$, *the following*:

$$L\phi(x) \approx \langle \gamma, \nabla\phi(x)\rangle + \frac{1}{2}\sum_{1\le i,j\le d} a_{i,j}\,\partial_{i,j}\phi(x) + \int_{{}^*\mathbb{R}^d}\left(\phi(x+y) - \phi(x) - \frac{\langle y, \nabla\phi(x)\rangle}{1+|y|^2}\right){}^*\nu(dy). \tag{1}$$

Moreover, note that both sides are finite.

Proof We begin by perturbing $\mathbb{A}$ infinitesimally and letting $\overline{\mathbb{A}} := \mathbb{A} + \iota\,\mathbb{I}$, where ι is any positive infinitesimal and $\mathbb{I}$ the identity matrix, so

$$\overline{\mathbb{A}} \quad \text{is } \textit{positive definite symmetric and} \quad \overline{\mathbb{A}} \approx \mathbb{A}. \tag{2}$$

We will show how to choose some ϵ, $\xi \in {}^*\mathbb{R}$ and α, $\kappa \in {}^*\mathbb{R}^d$ satisfying

$$|\kappa| < \epsilon < 2\epsilon < \xi \approx 0 \tag{3}$$

and certain additional properties and let η be the following sum of measures on ${}^*\mathbb{R}^d$ with disjoint supports:

$$\eta(dy) = \mathscr{N}[\alpha+\kappa;\, \Delta t\overline{\mathbb{A}}](y)\,\mathbf{1}_{|y-\kappa|\le\epsilon}(y)dy \;+\; \Delta t\,\mathbf{1}_{|y|\ge\xi}(y)\,{}^*\nu(dy), \tag{4}$$

where $\mathbf{1}_C$ denotes the indicator function of the set C and $\mathscr{N}$ the Gaussian density

$$\mathscr{N}[a;\sigma](y) := (2\pi)^{-\frac{d}{2}}(\det\sigma)^{-\frac{1}{2}}\,\exp\left(-\frac{1}{2}\langle y-a,\ \sigma^{-1}(y-a)\rangle\right),$$

where $a \in {}^*\mathbb{R}^d$ and σ is a $d\times d$ positive definite symmetric matrix (covariance matrix); then finally define the probability measure

$$\delta(dy) := \eta(dy)/\eta({}^*\mathbb{R}^d) \tag{5}$$

and prove that the associated L satisfies (1).

First of all we fix some ϵ so that

$$\epsilon \approx 0 \quad \text{and} \quad \epsilon\sqrt{N} \approx \infty. \tag{6}$$

(For example, take $\epsilon = N^{-1/3}$). Define for $n \in \mathbb{N}$ the internal sets

$$C_n := \left\{ r \in {}^*\mathbb{R} \,\middle|\, 2\epsilon < r < \frac{1}{n} \quad \text{and} \quad \frac{1}{r^2}\int_{|y|\ge r}\frac{|y|^2}{1+|y|^2}\,{}^*\nu(dy) < \sqrt{N}\right\}.$$

Each $C_n \neq \emptyset$ according to the assumption on ν, hence they form a decreasing sequence of nonempty internal sets and therefore, by the ω_1-saturation principle[2], we can find some $\xi \in \bigcap_{n\in\mathbb{N}} C_n$. So this ξ satisfies

$$0 \approx \xi > 2\epsilon \quad \text{and} \quad \frac{\sqrt{\Delta t}}{\xi^2}\int_{|y|\ge\xi}\frac{|y|^2}{1+|y|^2}\,{}^*\nu(dy) < 1. \tag{7}$$

Define

$$J_\xi := \int_{|y|\ge\xi}\frac{y}{1+|y|^2}\,{}^*\nu(dy) \in {}^*\mathbb{R}^d.$$

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[1] $|\big(\phi(x+y)-\phi(x)\big)(1+|y|^2) - \langle y, \nabla\phi(x)\rangle|/(1+|y|^2) = |\big(\phi(x+y)-\phi(x)\big)|y|^2 + \frac{1}{2}\langle y,\, \mathbb{H}\phi(\lambda_{x,y})\,y\rangle|/(1+|y|^2) \le C|y|^2/(1+|y|^2)$ for some $\lambda_{x,y}$ between x and $x+y$ and some constant C. Here $\mathbb{H}$ denotes $[\partial_{i,j}]_{1\le i,j\le d}$.

[2] Actually the *overspill* suffices, a principle which holds in any nonstandard construction.

Note that

$$|J_\xi| \,\le\, \int_{|y|\ge\xi} \frac{|y|}{1+|y|^2}\,{}^*\nu(dy) \,\le\, \frac{1}{\xi}\int_{|y|\ge\xi}\frac{|y|^2}{1+|y|^2}\,{}^*\nu(dy) \,<\, \xi\sqrt{N},$$

using the assumption on ν. In particular,

$$\sqrt{\Delta t}\,|J_\xi| \,\approx\, 0. \tag{8}$$

Next define

$$\alpha \,:=\, \Delta t\,(\gamma - J_\xi) \,\in\, {}^*\mathbb{R}^d. \tag{9}$$

By (8) we have

$$\sqrt{N}\,\alpha \,\approx\, 0. \tag{10}$$

The following results are from (6) and (10):

$$\begin{cases} \varrho \,:=\, \sqrt{N}\,\alpha \,-\, \displaystyle\int_{|y|\le\epsilon\sqrt{N}} y\,\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy \,\approx\, 0, \\ \\ \vartheta \,:=\, 1 \,-\, \displaystyle\int_{|y|\le\epsilon\sqrt{N}} \mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy \,\approx\, 0. \end{cases} \tag{11}$$

Define

$$\kappa \,:=\, \frac{\sqrt{\Delta t}}{1-\vartheta}\,\varrho \,\in\, {}^*\mathbb{R}^d, \tag{12}$$

then (11) implies that

$$\sqrt{N}\,\kappa \,\approx\, 0. \tag{13}$$

We remark that now (3) holds because of (6), (7) and (13); therefore, (4) and (5) define the measure η, as well as its normalization δ, as a sum of two measures on disjoint supports.

It is straightforward to use a change of variables to check the following for a given integrable $f:\,{}^*\mathbb{R}^d \to {}^*\mathbb{R}$:

$$\int_{|y-\kappa|\le\epsilon} f(y)\mathscr{N}\big[\alpha+\kappa;\,\Delta t\overline{\mathbb{A}}\big](y)\,dy \,=\, \int_{|y|\le\epsilon\sqrt{N}} f\big(\sqrt{\Delta t}\,y+\kappa\big)\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy,$$

i.e.

$$\int_{|y-\kappa|\le\epsilon} f(y)\,\eta(dy) \,=\, \int_{|y|\le\epsilon\sqrt{N}} f\big(\sqrt{\Delta t}\,y+\kappa\big)\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy. \tag{14}$$

Claim 1 $\eta({}^*\mathbb{R}^d) \,\approx\, 1.$

To prove the claim we write

$$\eta({}^*\mathbb{R}^d) \,=\, \int_{|y-\kappa|\le\epsilon}\eta(dy) \,+\, \Delta t\int_{\xi\le|y|\le 1}{}^*\nu(dy) \,+\, \Delta t\int_{|y|\ge 1}{}^*\nu(dy).$$

By (14) the first integral can be re-written as $\int_{|y|\le\epsilon\sqrt{N}}\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy$, which is ≈ 1 by (11).

Since $\frac{\xi^2}{2} \le \frac{|y|^2}{1+|y|^2}$ holds on the intervals $\xi \le |y| \le 1$, the second integral is $\le$ $\frac{2\Delta t}{\xi^2}\int_{\xi\le|y|\le 1}\frac{|y|^2}{1+|y|^2}\,{}^*\nu(dy)$, which is ≈ 0 by (7).

On the intervals $|y|\ge 1$ we have $1 \,\le\, \frac{2|y|^2}{1+|y|^2}$, so by the assumption on ν the third integral is also ≈ 0 and therefore the claim is proved.

Claim 2 $\frac{1}{\Delta t}\int_{|y-\kappa|\le\epsilon} y_i\,y_j\,\eta(dy) \,\approx\, a_{i,j},$ *where* $1\le i,j\le d$.

By (14), the left-hand side is

$$\int_{|y|\le\epsilon\sqrt{N}} (y_i + \sqrt{N}\,\kappa_i)(y_j + \sqrt{N}\,\kappa_j)\,\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy,$$

which by (6), (10) and (13) is $\approx \int_{|y|\le\epsilon\sqrt{N}} y_i\,y_j\,\mathscr{N}\big[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\big](y)\,dy$.

Then the claim follows from (6), (10) and (2).

We are now ready to prove (1). Without loss of generality, we assume that $\phi \in {}^*\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{R})$. Write

$$\begin{aligned}\eta({}^*\mathbb{R}^d)\,L\phi(x) &= \frac{1}{\Delta t}\int_{{}^*\mathbb{R}^d}(\phi(x+y)-\phi(x))\,\eta(dy)\\ &= \frac{1}{\Delta t}\int_{{}^*\mathbb{R}^d}(\phi(x+y)-\phi(x)-\langle y,\nabla\phi(x)\rangle)\,\eta(dy) + \frac{1}{\Delta t}\int_{{}^*\mathbb{R}^d}\langle y,\nabla\phi(x)\rangle\,\eta(dy)\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{cases} I_1 = \dfrac{1}{\Delta t}\displaystyle\int_{|y-\kappa|\le\epsilon}\frac{1}{2}\Big\langle y,\,\mathbb{H}\phi(\lambda_{x,y})\,y\Big\rangle\,\eta(dy),\\[2ex] I_2 = \dfrac{1}{\Delta t}\displaystyle\int_{|y-\kappa|\ge\epsilon}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\eta(dy),\\[2ex] I_3 = \dfrac{1}{\Delta t}\left(\displaystyle\int_{|y-\kappa|\ge\epsilon}\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\eta(dy) + \int_{|y-\kappa|\le\epsilon}\langle y,\nabla\phi(x)\rangle\,\eta(dy)\right),\end{cases}$$

with $\lambda_{x,y}$ denoting some number chosen from the interval between x and $x+y$ such that $\phi(x+y)-\phi(x)-\langle y,\nabla\phi(x)\rangle = \frac{1}{2}\big\langle y,\,\mathbb{H}\phi(\lambda_{x,y})\,y\big\rangle$, where the matrix $\mathbb{H}\phi$ is the Hessian $[\partial_{i,j}\phi]_{1\le i,j\le d}$.

Note that, for all $|y-\kappa|\le\epsilon$, $H_{x,y} := \mathbb{H}\phi(\lambda_{x,y}) - \mathbb{H}\phi(x) \approx 0$,

$$I_1 = \frac{1}{\Delta t}\int_{|y-\kappa|\le\epsilon}\frac{1}{2}\big\langle y,\,\mathbb{H}\phi(x)\,y\big\rangle\,\eta(dy) + \frac{1}{\Delta t}\int_{|y-\kappa|\le\epsilon}\frac{1}{2}\big\langle y,\,H_{x,y}\,y\big\rangle\,\eta(dy),$$

and therefore there are $h_{i,j}\approx 0$, $1\le i,j\le d$, so that

$$I_1 = \frac{1}{2}\sum_{1\le i,j\le d}\frac{\partial_{i,j}\phi(x)}{\Delta t}\int_{|y-\kappa|\le\epsilon}y_i\,y_j\,\eta(dy) + \sum_{1\le i,j\le d}\frac{h_{i,j}}{\Delta t}\int_{|y-\kappa|\le\epsilon}y_i\,y_j\,\eta(dy).$$

Consequently, by Claim 2,

$$I_1 \approx \frac{1}{2}\sum_{1\le i,j\le d}a_{i,j}\,\partial_{i,j}\phi(x). \tag{15}$$

From the definition of η we have

$$I_2 = \int_{|y|\ge\xi}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\,{}^*\nu(dy),$$

so

$$I_2 \approx \int_{{}^*\mathbb{R}^d}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\,{}^*\nu(dy). \tag{16}$$

The integral I_3 has the form $\langle\beta,\nabla\phi(x)\rangle$ for some

$$\beta := \frac{1}{\Delta t}\int_{|y-\kappa|\ge\epsilon}\frac{y}{1+|y|^2}\eta(dy) + \frac{1}{\Delta t}\int_{|y-\kappa|\le\epsilon}y\,\eta(dy).$$

The first term in β is simply J_ξ. For the second term, the i^{th} coordinate equals

$$\begin{aligned}
&\frac{1}{\Delta t}\int_{|y-\kappa|\le\epsilon} y_i\,\eta(dy)\\
&\quad= \frac{1}{\Delta t}\int_{|y|\le\epsilon\sqrt{N}} \bigl(\sqrt{\Delta t}\,y_i+\kappa_i\bigr)\mathscr{N}\bigl[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\bigr](y)\,dy \qquad\text{(by (14))}\\
&\quad= \frac{1}{\sqrt{\Delta t}}\int_{|y|\le\epsilon\sqrt{N}} y_i\,\mathscr{N}\bigl[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\bigr](y)\,dy + \frac{\kappa_i}{\Delta t}\int_{|y|\le\epsilon\sqrt{N}} \mathscr{N}\bigl[\sqrt{N}\,\alpha;\,\overline{\mathbb{A}}\bigr](y)\,dy\\
&\quad= \frac{1}{\sqrt{\Delta t}}\bigl(\sqrt{N}\,\alpha_i - \varrho_i\bigr) + \frac{\kappa_i}{\Delta t}\bigl(1-\vartheta\bigr) \qquad\text{(by (11))},
\end{aligned}$$

i.e.

$$\beta = J_\xi + N\alpha - \frac{\varrho}{\sqrt{\Delta t}} + \frac{\kappa}{\Delta t}\bigl(1-\vartheta\bigr),$$

which by (9) and (12) is just simplified as γ. Thus we have

$$I_3 = \langle\gamma,\nabla\phi(x)\rangle. \tag{17}$$

Finally, putting (15), (16), (17) and Claim 1 together, (1) is proved.

We also have the following converse of Theorem 1:

Theorem 2 *Let δ be an internal probability measure on ${}^*\mathbb{R}^d$ such that, for some $\xi\approx 0$, the following are finite:*

(a) $\frac{1}{\Delta t}\int_{|y|\ge\xi}\frac{|y|^2}{1+|y|^2}\delta(dy)$;

(b) $\frac{1}{\Delta t}\int_{|y|\le\xi} y_i\,y_j\,\delta(dy), \quad 1\le i,j\le d$;

(c) $\frac{1}{\Delta t}\bigl(\int_{|y|\ge\xi}\frac{y}{1+|y|^2}\delta(dy) + \int_{|y|\le\xi} y\,\delta(dy)\bigr)$.

Then there is a standard Lévy triple $(\gamma,\mathbb{A},\nu)$ for which the following holds for the generator L associated with δ and $\phi\in\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$, $x\in\mathbb{R}^d$:

$$L\phi(x) \approx \langle\gamma,\nabla\phi(x)\rangle + \frac{1}{2}\sum_{1\le i,j\le d} a_{i,j}\,\partial_{i,j}\phi(x) + \int_{\mathbb{R}^d}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\nu(dy).$$

Proof First define

$$\begin{aligned}
\gamma &:= {}^\circ\!\left(\frac{1}{\Delta t}\left(\int_{|y|\ge\xi}\frac{y}{1+|y|^2}\delta(dy) + \int_{|y|\le\xi} y\,\delta(dy)\right)\right),\\
\mathbb{A} &:= [a_{i,j}]_{1\le i,j\le d}, \quad\text{where}\quad a_{i,j} = {}^\circ\!\left(\frac{1}{\Delta t}\int_{|y|\le\xi} y_i\,y_j\,\delta(dy)\right),
\end{aligned}$$

and, for any Borel subset $S\subset\mathbb{R}^d$,

$$\nu(S) := \upsilon_L(S\setminus\{0\}), \quad\text{where}\quad \upsilon(dy) = \mathbf{1}_{|y|\ge\xi}(y)\frac{1}{\Delta t}\delta(dy).$$

It follows from (a) and the definition of ν that it is a standard Borel measure on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}\frac{|y|^2}{1+|y|^2}\nu(dy)<\infty$ and $\nu(\{0\})=0$. By (c), $\gamma\in\mathbb{R}^d$, and (b) shows that $\mathbb{A}$ is a standard $d\times d$ matrix. Also $\mathbb{A}$ is clearly nonnegative definite and symmetric. Therefore, $(\gamma,\mathbb{A},\nu)$ defines a standard Lévy triple.

Let L be the generator associated with δ. Let $\phi\in\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C})$ and $x\in\mathbb{R}^d$. Then similarly to the proof of Theorem 1, we have the decomposition

$$L\phi(x) = I_1 + I_2 + I_3,$$

where

$$\begin{cases} I_1 &= \dfrac{1}{\Delta t}\displaystyle\int_{|y|\le\xi} \frac{1}{2}\Big\langle y,\, \mathbb{H}\phi(\lambda_{x,y})\, y\Big\rangle\, \delta(dy), \\[2ex] I_2 &= \dfrac{1}{\Delta t}\displaystyle\int_{|y|\ge\xi} \left(\phi(x+y)-\phi(x)-\frac{\langle y, \nabla\phi(x)\rangle}{1+|y|^2}\right)\delta(dy), \\[2ex] I_3 &= \dfrac{1}{\Delta t}\left(\displaystyle\int_{|y|\ge\xi} \frac{\langle y, \nabla\phi(x)\rangle}{1+|y|^2}\delta(dy) \;+\; \int_{|y|\le\xi}\langle y, \nabla\phi(x)\rangle\,\delta(dy)\right), \end{cases}$$

where $\lambda_{x,y}$ as given before.

Once again, the same computation shows that

$$I_1 \approx \frac{1}{2}\sum_{1\le i,j\le d} a_{i,j}\,\partial_{i,j}\phi(x).$$

On the other hand, by the assumptions on ϕ it follows that

$$\int_{\mathbb{R}^d}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\nu(dy) \approx \frac{1}{\Delta t}\int_{|y|\ge\xi}\left(\phi(x+y)-\phi(x)-\frac{\langle y,\nabla\phi(x)\rangle}{1+|y|^2}\right)\delta(dy) = I_2.$$

Finally, from the definition of γ,

$$I_3 \approx \langle \gamma, \nabla\phi(x)\rangle,$$

hence the conclusion follows.

Remark 3 (i) From Theorem 2 we note that the δ in Theorem 1 corresponding to a given standard Lévy triple is not unique.

(ii) For $d=1$, notable cases of δ satisfying assumptions in Theorem 2 include:

- Brownian motion (see [3]): $\delta(dy) = \mathscr{N}[0;\Delta t](y)\,dy$ and $\xi = \sqrt{\Delta t}$. The standard parts of terms in (a)-(c) are:

$$1-\frac{211}{420\sqrt{2\pi}};\quad \frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{2}} e^{-x^2}dx - \sqrt{\frac{2}{\pi e}} = 0.1987\cdots;\quad 0.$$

- Poisson process of finite intensity $a>0$:

$\delta(dy) = (1-a\Delta t)\mathscr{N}[0;\Delta t^2](y)\,dy + a\Delta t\mathscr{N}[1;\Delta t^2](y)\,dy$ and $\xi = \Delta t$. For $a=1$, the standard parts of terms in (a)–(c) are: 0; 0; 0.

- Cauchy distribution: $\delta(dy) = \frac{\Delta t}{\pi(y^2+\Delta t^2)}dy$ and $\xi = \Delta t$. The standard parts of terms in (a)–(c) are: 1; $\frac{2}{\pi}-\frac{1}{2}$; 0.

(iii) Given an internal probability measure δ corresponding to a standard Lévy triple as in Theorem 1, the proof of the theorem shows that, for any $\phi\in\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{R})$, the function $L\phi : {}^*\mathbb{R}^d \to {}^*\mathbb{R}$ is S-bounded and has a continuous standard part. If, in addition, $\mathbb{H}\phi$ is uniformly continuous then (1) holds for all $x\in{}^*\mathbb{R}$. If all ϕ, $\nabla\phi$, and $\mathbb{H}\phi$ are uniformly continuous, then $L\phi$ is S-continuous.

(iv) If we define a semigroup $\{T_t\}_{t\in\mathbb{T}}$ by $T_t := T_{\Delta t}^{tN}$, where

$$T_{\Delta t}\phi(x) := {}^\circ\!\int_{{}^*\mathbb{R}^d} {}^*\phi(x+y)\delta(dy),\quad \phi\in\mathscr{C}_b^2(\mathbb{R}^d;\mathbb{C}),$$

then L is the nonstandard version of an infinitesimal generator of this semigroup.

4 Lévy Processes

In this section we show by a calculation of the characteristic function using Theorem 1 that any arbitrary standard Lévy process in $\mathbb{R}^d$ equals in law a hyperreal random walk, i.e. a hyperfinite sum of hyperreal increments (Proposition 12). The finiteness and stochastic continuity of the sum are also consequences of Theorem 1. The result will be improved in the next section by showing that it is in fact a càdlàg version.

We assume throughout this section an internal probability measure δ satisfying (1) in Theorem 1 for some standard Lévy triple $(\gamma, \mathbb{A}, \nu)$. We work with the product space $\Omega := ({}^*\mathbb{R}^d)^{\mathbb{T}}$ under the product probability $\mu := \delta^{\mathbb{T}}$. We emphasize that the space is not hyperfinite but the timeline is. Sample paths in Ω are written as $\omega = \{\omega_t\}_{t\in\mathbb{T}}$. Let's consider the following:

Definition 4 *We let* $X : \Omega\times\mathbb{T} \to {}^*\mathbb{R}^d$ *be the internal process given by the hyperfinite sum of hyperreal increments, i.e.*

$$X_t(\omega) := \sum_{s<t}\omega_s, \tag{18}$$

and by default we set $X_0 \equiv 0$.

We now prove the Lévy–Khintchine formula for the characteristic function of μ :

Theorem 5 *Let* $t \in \mathbb{T}$ *be finite and* $u \in \mathbb{R}^d$. *Then*

$$\mathbb{E}\left[e^{i\langle u, X_t\rangle}\right] \approx \exp\left[t\left(i\langle\gamma, u\rangle - \frac{1}{2}\langle u, \mathbb{A}u\rangle + \int_{\mathbb{R}}\left(e^{i\langle u,y\rangle} - 1 - \frac{i\langle u, y\rangle}{1+|y|^2}\right)\nu(dy)\right)\right]. \tag{19}$$

Proof We first apply Theorem 1 to

$$\phi(y) = e^{i\langle u,y\rangle} \quad\text{and}\quad x = 0 :$$

$$\begin{aligned}\mathbb{E}\left[e^{i\langle u,X_t\rangle}\right] &= \mathbb{E}\left[\prod_{s<t} e^{i\langle u,\omega_s\rangle}\right] = \left(\int_{{}^*\mathbb{R}^d} e^{i\langle u,y\rangle}\delta(dy)\right)^{tN} = (1 + L\phi(0)\,\Delta t)^{tN} \\ &= \Bigg(1 + \frac{1}{N}\Bigg[\iota + \langle\gamma, \nabla\phi(0)\rangle + \frac{1}{2}\sum_{1\le k,j\le d} a_{k,j}\,\partial_{k,j}\phi(0) \\ &\qquad + \int_{\mathbb{R}^d}\left(\phi(y) - \phi(0) - \frac{\langle y, \nabla\phi(0)\rangle}{1+|y|^2}\right)\nu(dy)\Bigg]\Bigg)^{tN} \\ &= \left(1 + \frac{1}{N}\left[\iota + i\langle\gamma, u\rangle - \frac{1}{2}\langle u, \mathbb{A}u\rangle + \int_{\mathbb{R}}\left(e^{i\langle u,y\rangle} - 1 - \frac{i\langle u, y\rangle}{1+|y|^2}\right)\nu(dy)\right]\right)^{tN},\end{aligned}$$

where ι is some infinitesimal and we use $\phi(0) = 1$, $\nabla\phi(0) = i\,u$ and $\partial_{k,j}\phi(0) = -u_k u_j$ in the last line. Hence (19) is proved.

Definition 6 *Let's define the following approximation of the sample space* Ω :

$$\Omega_{\tau,n} := ([-n, n]^d)^{\mathbb{T}\cap[0,\tau)} \times ({}^*\mathbb{R}^d)^{\mathbb{T}\cap[\tau,\infty)} \subset \Omega, \qquad \tau \in \mathbb{T},\ n \in \mathbb{N}. \tag{20}$$

Lemma 7 *Let* $\tau \in \mathbb{T}$ *be finite. Then* $\mu_L(\bigcup_{n\in\mathbb{N}}\Omega_{\tau,n}) = 1$.

Proof The condition $\int_{\mathbb{R}^d}\frac{|y|^2}{1+|y|^2}\nu(dy) < \infty$ implies that

$$\int_{|y|\ge n}\nu(dy) \to 0 \quad\text{as}\quad n\to\infty \quad\text{in}\quad \mathbb{N}.$$

For $n \ge 1$,

$$\mu(\Omega_{\tau,n}) \ge \left(1 - \int_{|y|\ge n}\delta(dy)\right)^{\tau N} = \left(1 - \Delta t\int_{|y|\ge n}\nu(dy)\right)^{\tau N} \approx \exp\left(-\tau\int_{|y|\ge n}\nu(dy)\right),$$

hence $\mu_L(\Omega_{\tau,n}) \to 1$ as $n \to \infty$ in $\mathbb{N}$, so the conclusion holds.

Lemma 8 *Let* $\tau \in \mathbb{T}$ *be finite,* $n \in \mathbb{N}$ *and* $p \in \mathbb{Z}$. *Then, for each* $1 \le i \le d$, *there is some* $C_{i,p} \in {}^*\mathbb{R}$ *such that* $0 < C_{i,p} < \infty$ *and*

$$\mathbb{E}_{\Omega_{\tau,n}}\left[e^{p\,(X_\tau)_i}\right] \le C_{i,p}^{\tau}.$$

Proof Apply Theorem 1 to some nonnegative $\phi \in \mathscr{C}_b^2(\mathbb{R}^d;\mathbb{R})$ such that $\phi(y) = e^{p\,y_i}$ on $[-n,n]^d$. Then

$$\mathbb{E}_{\Omega_{\tau,n}}[e^{p\,(X_\tau)_i}] = \Big(\int_{[-n,n]^d} e^{p\,y_i}\delta(dy)\Big)^{\tau N} \le \Big(\int_{{}^*\mathbb{R}^d} \phi(y)\delta(dy)\Big)^{\tau N} = \big(1 + L\phi(0)\Delta t\big)^{\tau N}.$$

Now let

$$C_{i,p} := (1 + L\phi(0)\Delta t)^N.$$

Then $C_{i,p} \approx e^{L\phi(0)} < \infty$ by Theorem 1.

Corollary 9 *For every finite $t \in \mathbb{T}$, X_t is a.s. finite with respect to the Loeb measure μ_L.*

Proof By Lemma 8, for $n \in \mathbb{N}$ and $1 \le i \le d$, we have both $\mathbb{E}_{\Omega_{t,n}}\left[e^{(X_t)_i}\right] < \infty$ and $\mathbb{E}_{\Omega_{t,n}}\left[e^{-(X_t)_i}\right] < \infty$, so $(X_t)_i$ must be *a.s.* finite on $\Omega_{t,n}$. Then the conclusion follows from Lemma 7.

Corollary 10 *Let $t \approx 0$ in $\mathbb{T}$. Then $\mu_L\big(X_t \approx 0\big) = 1$.*

Proof By Lemma 8, for any $n \in \mathbb{N}$, $p \in \mathbb{Z}$ and $1 \le i \le d$, $\mathbb{E}_{\Omega_{t,n}}[e^{p\,(X_t)_i}] \le C_{i,p}^t \approx 1$. Since this holds for all $p \in \mathbb{Z}$ we have $(X_t)_i \approx 0$ a.s. on $\Omega_{t,n}$ and the conclusion follows from Lemma 7.

Now we define the standard part of X and will show that it has the expected properties.

Definition 11 *For each $t \in [0,\infty)$ we let $\bar{t} := \min\{s \in \mathbb{T} \,|\, s \ge t\}$ and define*

$$x : \Omega \times [0,\infty) \to \mathbb{R}^d \quad \text{by} \quad x_t(\omega) := {}^\circ X_{\bar{t}}(\omega). \tag{21}$$

By Corollary 9, x_t is *a.s.* finite with respect to μ_L for all $t \in [0,\infty)$, so x is a *standard stochastic process* under μ_L.

By Corollary 10, for each $r \in \mathbb{R}^+$ and finite $s,\, t \in \mathbb{T}$ with $t \approx 0$ we have $\mu_L\big(|X_{s+t} - X_s| > r\big) = \mu_L\big(|X_t| > r\big) = 0$. Hence, for $s,\, t \in [0,\infty)$,

$$\mu_L\big(|x_{s+t} - x_s| > r\big) \to 0 \quad \text{as} \quad t \to 0,$$

i.e. x is *stochastically continuous.*

Also by the construction of X, x is clearly *time-homogeneous*, has *independent increments* and $x_0 \equiv 0$.

That is, x is a *Lévy process in law.*

Moreover, for each fixed $u \in \mathbb{R}^d$, the function $y \mapsto e^{i\langle u,y\rangle}$ is an S-bounded function on ${}^*\mathbb{R}^d$, so Lemma 8 and Corollary 9 imply that, for each $t \in [0,\infty)$,

$$\exp\big(i\langle u, X_{\bar{t}}\rangle\big) \quad \text{is an } SL^p\text{-lifting of} \quad \exp\big(i\langle u, x_t\rangle\big), \quad p \in \mathbb{N}.$$

In particular,

$$\mathbb{E}_\mu\left[\exp\big(i\langle u, X_{\bar{t}}\rangle\big)\right] \approx \mathbb{E}_{\mu_L}\left[\exp\big(i\langle u, x_t\rangle\big)\right], \quad u \in \mathbb{R}^d,\ t \in [0,\infty);$$

therefore, by Theorem 5, x is a Lévy process in law corresponding to the triple $(\gamma, \mathbb{A}, \nu)$. Since each standard Lévy process is uniquely specified in law by a Lévy triple we have, from Theorem 1, the following summary of our representation result via internal processes. A stronger version of this will appear in the next section.

Proposition 12 *Let $\bar{x} : \Omega \times [0,\infty) \to \mathbb{R}^d$ be an arbitrary Lévy process in law. Then it is equal in law to x, the stochastic process obtained from the hyperfinite sum in Definition* 4 *and* 11 *for the Lévy triple of $\bar{x}$.*

5 Martingales and Càdlàg

In this section we improve Proposition 12 by showing in Theorem 18 that the standard part x is in fact càdlàg. This is done by applying consequences of Theorem 1 to show that, on an approximating set, the martingale part of X is SL^2, and then use nonstandard martingale theory.

First we need some estimate, again a consequence of Theorem 1.

Lemma 13 *Let $n \in \mathbb{N}$. Then*

$$\left| \frac{1}{\Delta t} \int_{|y| \le n} y_i \, \delta(dy) \right| < \infty, \quad 1 \le i \le d.$$

Proof Only $n \ge 1$ needs to be considered. On $|y| \ge n$, we have $\delta = \Delta t \, {}^*\nu$. The assumption on ν implies that $c_n := \nu\big(|y| \ge n\big) < \infty$. So there is $\phi \in \mathscr{C}_b^2(\mathbb{R}^d; \mathbb{R})$ such that $\phi(y) = y_i$ on $|y| \le n$ and

$$\left| \int_{|y| \ge n} \phi(y) \, \delta(dy) \right| \le 2nc_n \Delta t.$$

Then

$$\left| \frac{1}{\Delta t} \int_{|y| \le n} y_i \, \delta(dy) \right| \le \left| \int_{{}^*\mathbb{R}^d} \frac{\phi(y)}{\Delta t} \, \delta(dy) \right| + 2nc_n.$$

Note that the right side equals $|L\phi(0)| + 2nc_n$, which is finite by Theorem 1.

Let $\{\mathscr{F}_t\}_{t \in \mathbb{T}}$ be the *internal filtration* generated by $\{X_t\}_{t \in \mathbb{T}}$ in Definition 4. If we let

$$M_t := X_t - \frac{t}{\Delta t} \int_{{}^*\mathbb{R}^d} y \delta(dy) \tag{22}$$

then $\{M_t\}_{t \in \mathbb{T}}$ is an *internal martingale* with respect to $\{\mathscr{F}_t\}_{t \in \mathbb{T}}$ since

$$\mathbb{E}\left[M_{t+\Delta t} - M_t \middle| \mathscr{F}_t\right] = \int_{{}^*\mathbb{R}^d} y\delta(dy) - \int_{{}^*\mathbb{R}^d} y\delta(dy) = 0.$$

The problem is that the right-hand side of (22) need not be finite, so we require the following approximation:

Definition 14 *Let $\tau \in \mathbb{T}$ and $n \in \mathbb{N}$. Let $\Omega_{\tau,n}$ be as in Definition 6. Then for $t < \tau$, we define*

$$M_t^{\tau,n} := X_t \, \mathbf{1}_{\Omega_{\tau,n}} - \frac{t}{\Delta t} \int_{y \in [-n,n]^d} y \, \delta(dy). \tag{23}$$

Similarly to the above, $\{M_t^{\tau,n}\}_{t \in \mathbb{T}}$ is an *internal martingale.* But there is more:

Lemma 15 *For all $t < \tau$, $\tau \in \mathbb{T}$ finite, we have $\mathbb{E}[|M_t^{\tau,n}|^2] < \infty$.*

Proof We have, for each $1 \le i \le d$,

$$\mathbb{E}_{\Omega_{\tau,n}}\left[\exp\big((M_t^{\tau,n})_i\big)\right] = \mathbb{E}_{\Omega_{\tau,n}}\left[\exp\big((X_t)_i\big)\right] \exp\left(- \frac{t}{\Delta t} \int_{y \in [-n,n]^d} y_i \, \delta(dy) \right) < \infty,$$

by Lemma 8 and 13. Similarly, $\mathbb{E}_{\Omega_{\tau,n}}\left[\exp\big(- (M_t^{\tau,n})_i\big)\right] < \infty$. So it follows that $\mathbb{E}_{\Omega_{\tau,n}}[\exp(|(M_t^{\tau,n})_i|)] < \infty$. But $2e^{|u|} \ge u^2$ and $\mathbb{E}[(M_t^{\tau,n})_i^2] = \mathbb{E}_{\Omega_{\tau,n}}[(M_t^{\tau,n})_i^2]$, the lemma is proved.

Remark 16 In [10] such $\{M_t^{\tau,n}\}_{t<\tau}$ above is called a λ^2-*martingale.* Moreover, by a proof similar to that of Lemma 15, $\{M_t^{\tau,n}\}_{t<\tau}$ is SL^p for all $p \in \mathbb{N}$. In particular, on $\Omega_{\tau,n}$ the standard process x given by Definition 11 has the *standard martingale* part given by

$\{{}^\circ M_{\bar{t}}^{\tau,n}\}_{t\in[0,\,{}^\circ\tau)}$ with respect to the filtration generated under the Loeb measure, i.e. the filtration of $\{x_t\}_{t\in[0,\,{}^\circ\tau)}$ restricted to $\Omega_{\tau,n}$.

We quote the following definition from [10]:

Definition 17 ([10] 4.2.9) *An internal function* $F : \mathbb{T} \to {}^*\mathbb{R}^d$ *is said to have the S-right limit* $r \in \mathbb{R}^d$ *at some* $t \in [0,\infty)$ *if for all* $\epsilon \in \mathbb{R}^+$ *there exists* $\delta \in \mathbb{R}^+$ *such that, whenever* $s \in \mathbb{T}$ *and* $t < {}^\circ s < t+\delta$, *we have* $|F(s) - r| < \epsilon$. *The S-left limit is defined similarly.*

Using Lemma 15, we can apply [10, Proposition 4.2.10] to conclude that a.s. $\{M_t^{\tau,n}\}_{t<\tau}$, hence also $\{X_t\,\mathbf{1}_{\Omega_{\tau,n}}\}_{t<\tau}$ has both the S-right limit and S-left limit. Since $n \in \mathbb{N}$ is arbitrary, it follows from Lemma 7 that the same holds for $\{X_t\}_{t<\tau}$ a.s. Consequently, by the way it is defined in Definition 11, $\{x_t\}_{t\in[0,\,{}^\circ\tau)}$, the standard part of $\{X_t\}_{t<\tau}$, is a.s. right continuous with left limits. Note that τ is an arbitrary finite element in $\mathbb{T}$.

The above are summarized as the following representation result:

Theorem 18 *Let* $\bar{x} : \Omega\times[0,\infty) \to \mathbb{R}^d$ *be an arbitrary Lévy process in law. Then there is an internal probability measure* δ *on* ${}^*\mathbb{R}^d$ *so that under the Loeb measure* μ_L *on* $\Omega = ({}^*\mathbb{R}^d)^{\mathbb{T}}$, *where* $\mu = \delta^{\mathbb{T}}$, *the standard Lévy process* x *given by*

$$x_t(\omega) = {}^\circ\sum_{s<\bar{t}} \omega_s, \quad \textit{where} \quad \bar{t} = \min\{s\in\mathbb{T} \mid s \geq t\},$$

is càdlàg equal to $\bar{x}$ *in law.*

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