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MARGIN TRADING THROUGH HYPER TIMELINE

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We consider a model of margin trading based on the hyperfinite timeline. Using only elementary nonstandard analysis we are able to derive explicit formulas for the expected margin call time and loss. Further margin trading strategy is studied and an application to pricing barrier option is given. We prove a generalization of the Catalan numbers which forms the combinatoric basis of our results and should be of independent interest.

Keywords: Nonstandard analysis; infinitesimal; hyperreal; hypermodel; Catalan numbers; margin trading; option pricing.

1. Introduction

In a margin trading, a trader must give his initial deposit P, referred to as a *margin*, to a broker as a collateral. Then the trader can enter a position on credits and start trading a basket of risky assets such as stocks and currencies. Let us denote the time t price of this basket of assets by S_t. According to the leverage structure $\lambda : 1$ the trader can enter a position of value λP and hence $\lambda P = S_0$, the present price of the basket of assets. With higher leverage one has access to higher potential profit but of course there is a correspondingly greater chance of incurring larger loss. Normally the leverage is something like $2 : 1$ in a stock market but it can be as high as $100 : 1$ in a currency market, depending on the acceptable level of risk. As a collateral requirement, the broker fix some constant $\gamma \in (0, 1)$, the margin requirement, i.e., margin call occurs when the asset value is below the requirement, meaning that the position will be closed automatically when the loss $S_0 - S_t$ first become $> \gamma P$.

Let $L \in (0, 1]$ represent the margin call level, as a percentage of the initial position S_0, That is,

$$LS_0 = S_0 - \gamma P, \quad \text{hence} \quad L = 1 - \frac{\gamma}{\lambda} \quad \text{and} \quad \lambda = \frac{\gamma}{1 - L}.$$

We also get from here that

$$P = \frac{(1 - L)\, S_0}{\gamma}.$$

To put the above in another way, a margin call occurs at time t when S_t dropped below $L\, S_0$ for the first time.

Other than a margin call taking place, we assume that the trader holds onto the position throughout the time horizon $[0, T]$. In order to assess the risk involved in such trading, one needs to know within the time horizon $[0, T]$ things like the average time of margin call, the average loss, the maximum loss and the most dangerous margin level to avoid.

In this paper we will answer these questions and derive explicit formulas, then we also consider a persistent strategy in margin trading and an application to pricing a barrier option. Our basic framework is that of hypermodels, namely tools from elementary nonstandard analysis and the hyper timeline, a discrete timeline constructed using a time step of infinitesimal size. The only nonstandard techniques we need are hyperreal arithmetic and hyperfinite Riemann sum. No saturation principle nor Loeb theory is ever needed. Indeed, there is no explicit reference to neither σ-measure theory nor Brownian motion. Our major tool from combinatorics involves counting some paths on the binomial tree, and in particular a generalization of the Catalan numbers is required. In principle, one can also derives results in this paper using standard stochastic methods such as the first hitting time formulas for geometric Brownian motion (see, for example, [2]) or the image methods from PDE, but our approach here is more elementary, intuitive and direct; it also illuminates the underlying combinatorial nature of the model.

The paper is organized as follows. In Sec. 2, we give a quick review of some notions from nonstandard analysis and the binomial tree we used. In Sec. 3 we first prove a lemma which forms the main technical tool for many calculations. As a consequence, we give an explicit formula for the expected margin call time within $[0, T]$. In Sec. 4 we derive a formula for the expected margin loss and its derivative with respect to the margin level L. This is useful for finding the most disadvantageous level. In Sec. 5 we introduce the persistent strategy, in which the trader continues margin trading with whatever remains, regardless of the loss. Somewhat surprisingly it appears that the persistent strategy does not lead to an eventual total loss. In Sec. 6 we use another view of the margin call property and give an application producing an explicit formula for pricing the down knock-in barrier option. Finally in the Appendix we prove a generalization of the Catalan numbers which is the main combinatorial tool in counting paths corresponding to margin calls in the main lemma in Sec. 3.

This paper can be regarded as part of the on-going program starting with [6] and [7] emphasizing the direct and intuitive tools of hypermodels and the conviction that in modelling financial trading the hyper timeline is the correct notion.

2. Preliminaries

We first begin with a brief summary of some notions from hypermodels. In nonstandard analysis, the nonstandard extension of a standard object X is denoted by ${}^*X$. Infinite elements in the set ${}^*\mathbb{N}$ of nonstandard natural numbers are called *hyperfinite*; a set counted internally by a hyperfinite number is also called hyperfinite; given $r, s \in {}^*\mathbb{R}$, hyperreal numbers, if $|r - s| < q$ for all $q \in \mathbb{R}^+$, we write $r \approx s$ (*infinitely close*); r is called *infinitesimal* when $r \approx 0$; a finite element r of ${}^*\mathbb{R}$ ($r < \infty$) is one with $|r| < n$ for some $n \in \mathbb{N}$; such r is $\approx s$ for a unique $s \in \mathbb{R}$ (called the *standard part*, in symbol: $s = {}^\circ r$); write $r \approx \infty$ when r is infinite.

We take the present time to be 0 and without loss of generality we take the terminal time to be 1. The hyperfinite timeline is defined as

$$\mathbb{T} := \{0, \Delta t, 2\Delta t, \ldots, 1\},$$

where $\Delta t = 1/N$ and $N \in {}^*\mathbb{N}$ is a fixed hyperfinite number.

An internal function $F : \mathbb{T} \to {}^*\mathbb{R}$ is called S-continuous if $|F(t)| < \infty$, $t \in \mathbb{T}$, (since $\mathbb{T}$ is hyperfinite, this is equivalent to F finitely bounded) and $F(s) \approx F(t)$ whenever $s \approx t$. If F is S-continuous, there is a unique continuous $f : [0, 1] \to \mathbb{R}$ such that $f(u) = {}^\circ(F(t))$ whenever $t \approx u$. We write in this case $f = {}^\circ F$.

Further background on nonstandard analysis can be found in [1, 5, 6].

We will consider the market with a risky asset represented by S; for example, a single item or a portfolio of stocks or currencies. Let r denote the expected return rate of the asset; so it is also the riskless interest rate ρ in a risk-neutral environment. Let σ denote the positive non-infinitesimal volatility of the asset return rate.

By using S_0 as the numéraire, we can sometimes make the simplification that $S_0 = 1$.

Our model for the asset price S_t is given by a hyperfinite version of the Cox–Ross–Rubinstein centered binomial tree in which the up ratio is $u = e^{\sigma\sqrt{\Delta t}}$ and the down ratio is $u^{-1} = e^{-\sigma\sqrt{\Delta t}}$, i.e., they satisfies the centering condition. (See [3] for a variant of this hyperfinite binomial tree.) Therefore we take as sample space $\Omega = \{-1, +1\}^{\mathbb{T}}$ and for $\omega \in \Omega$

$$S_{t+\Delta t}(\omega) = S_t(\omega) e^{\omega_t \sigma\sqrt{\Delta t}} = S_t(\omega) e^{\pm\sigma\sqrt{\Delta t}}.$$

We let constant p denote the up transitional probability at each time t. Since r is the expected return rate of the asset, the following is satisfied:

$$\mathbb{E}_t \left[S_{t+\Delta t}\right] = S_t \, e^{r\Delta t},$$

i.e., $p\, u + (1 - p)\, u^{-1} = e^{r\Delta t}$, or

$$p = \frac{e^{r\Delta t} - u^{-1}}{u - u^{-1}} = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

We remark that since $\sigma > 0$ is finite, non-infinitesimal and $\Delta t \approx 0$, it follows that

$$p = \frac{1}{2} + c\sqrt{\Delta t}, \quad 1 - p = \frac{1}{2} + c'\sqrt{\Delta t}$$

for some finite c and c'; in particular, both p and $(1 - p)$ are $\approx \frac{1}{2}$.

3. Average Margin Call Time

Suppose we set a constant $0 < L \leq 1$ representing the margin call level, as a percentage of the present price S_0. Then a margin call occurs at time $t + \Delta t$ if

$$S_r(\omega) \geq L\, S_0 \quad \text{for all} \quad 0 \leq r \leq t, \quad \text{but} \quad S_{t+\Delta t}(\omega) < L\, S_0.$$

We let $\mathbf{1}_{MC}\,(\omega, t)$ denote the indicator function of a margin call at time $t + \Delta t$, i.e.,

$$\mathbf{1}_{MC}\,(\omega, t) = \begin{cases} 1 & \text{if } S_r(\omega) \geq L\, S_0 \quad \text{for all} \quad 0 \leq r \leq t \quad \& \quad S_{t+\Delta t}(\omega) < L\, S_0, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, $\mathbf{1}_{MC}\,(\omega, t)$ gives the first touch of the level $L\, u^{-1}\, S_0$.

The following is the main technical lemma that we need throughout the paper.

Lemma 3.1. *Suppose* $L = u^{-2m}$ *for some* $m \in {}^*\mathbb{N}$.

Let $F : \mathbb{T} \to {}^*\mathbb{R}$ *be S-continuous and define* $f = {}^\circ F$, *then*

$$\sum_{t \in \mathbb{T}} \mathbb{E}\left[F(t)\, \mathbf{1}_{MC}\,(\omega, t)\right] \approx C \int_0^1 \frac{f(t^2)}{t^2}\, e^{-\frac{A}{t^2} - Bt^2} dt,$$

where in the standard integral A, B, C *are the standard part of the constants*

$$\frac{1}{2}\left(\frac{\ln L}{\sigma}\right)^2, \quad \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 \quad \textit{and} \quad -\sqrt{\frac{2}{\pi}}\left(\frac{\ln L}{\sigma}\right) L^{\frac{r}{\sigma^2} - \frac{1}{2}}.$$

Proof. First note that $\sum_{t \in \mathbb{T}} \mathbb{E}\left[F(t)\, \mathbf{1}_{MC}\,(\omega, t)\right]$ is finite and we will implicitly use the finiteness of this and other sums.

With L of the form u^{-2m} i.e., $e^{-2m\sigma\sqrt{\Delta t}}$, margin call only occurs at odd times, i.e., time of the form $(2n + 1)\Delta t$, hence

$$\sum_{t \in \mathbb{T}} \mathbb{E}\left[F(t)\, \mathbf{1}_{MC}\,(\omega, t)\right] = \sum_{0 \leq n < \frac{N}{2}} \mathbb{E}\left[F(2n\Delta t)\, \mathbf{1}_{MC}\,(\omega,\, 2n\Delta t)\right]. \tag{3.1}$$

For a fixed $n \in {}^*\mathbb{N}$ the paths ω on which a margin call occurs at time $(2n+1)\Delta t$ are precisely those having the property that

$$\sum_{0 \leq i < k} \omega_{i\Delta t} \geq -2m, \quad 0 \leq k < 2n, \quad \sum_{0 \leq i < 2n} \omega_{i\Delta t} = -2m \quad \text{and} \quad \omega_{2n\Delta t} = -1.$$

By Theorem A.1 in the Appendix, the number of such paths $\{\omega_{i\Delta t}\}_{0 \leq i < 2n}$ is $\mathcal{C}_{n,m}$ and so (3.1) is

$$\begin{aligned} &\sum_{t \in \mathbb{T}} \mathbb{E}\left[F(t)\, \mathbf{1}_{MC}\,(\omega, t)\right] \\ &\quad = \sum_{0 \leq n < \frac{N}{2}} F(2n\Delta t)\, \mathcal{C}_{n,m}\, p^{n-m}(1-p)^{n+m+1} \end{aligned}$$

$$= (1-p) \sum_{0 \le n < \frac{N}{2}} F(2n\Delta t) \left(\frac{2m+1}{n+m+1}\right) \binom{2n}{n+m} (p\,(1-p))^n \left(\frac{1-p}{p}\right)^m$$

$$\approx \frac{1}{2} \sum_{0 \le n < \frac{N}{2}} F(2n\Delta t) \left(\frac{2m+1}{n+m+1}\right) \binom{2n}{n+m} (p\,(1-p))^n \left(\frac{1-p}{p}\right)^m. \tag{3.2}$$

We need to apply the following form of the Stirling's formula

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{\epsilon}{k}} \quad \text{for some} \quad 0 < \epsilon < \frac{1}{12},$$

and obtain

$$\binom{2n}{n+m} = \sqrt{\frac{n}{\pi}}\,\frac{1}{\sqrt{n^2-m^2}}\,\frac{(2n)^{2n}}{(n+m)^{n+m}\,(n-m)^{n-m}}\,\lambda_{n,m}$$

$$= \frac{1}{\sqrt{\pi n}}\left(1-\left(\frac{m}{n}\right)^2\right)^{-\frac{1}{2}} \left(\frac{n^{2n}}{(n+m)^{n+m}\,(n-m)^{n-m}}\right) 2^{2n}\,\lambda_{n,m},$$

for some $\lambda_{n,m}$ which is always finite and is ≈ 1 whenever both n and $n-m$ are infinite.

Now we fix some $M \in {}^*\mathbb{N}$ such that $M\sqrt{\Delta t} \approx \infty$ but $M\Delta t \approx 0$. (For example, take $M = N^{2/3}$.) Since $L = e^{-2m\sigma\sqrt{\Delta t}}$ is finite, $m\sqrt{\Delta t}$ has to be finite, so $n-m$ is infinite and $\frac{m}{n} \approx 0$ whenever $n \ge M$.

Since $\sum_{0 \le t < 2M\Delta t} \mathbb{E}\left[F(t)\,1_{MC}\,(\omega,t)\right] \approx 0$, we can sum (3.2) from M instead of 0 and (3.2), hence (3.1) is infinitely close to

$$\frac{1}{2\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{\sqrt{n}} \left(\frac{2m+1}{n+m+1}\right) \left(\frac{n^{2n}}{(n+m)^{n+m}\,(n-m)^{n-m}}\right)$$

$$\times\,(4p\,(1-p))^n \left(\frac{1-p}{p}\right)^m. \tag{3.3}$$

Now for $n \ge M$,

$$\frac{n^{2n}}{(n+m)^{n+m}\,(n-m)^{n-m}} = \left(\frac{n^2}{n^2-m^2}\right)^n \left(1-\frac{2m}{n+m}\right)^m$$

$$= \left(1+\frac{m^2\Delta t}{(n^2-m^2)\Delta t}\right)^{(n^2-m^2)\Delta t\,\frac{n\Delta t}{(n^2-m^2)(\Delta t)^2}}$$

$$\times \left(1-\frac{2m\sqrt{\Delta t}}{(n+m)\sqrt{\Delta t}}\right)^{(n+m)\sqrt{\Delta t}\,\frac{m\sqrt{\Delta t}}{(n+m)\Delta t}}$$

$$\approx e^{m^2\Delta t\,\frac{1}{n\Delta t}}\,e^{-2m\sqrt{\Delta t}\,\frac{m\sqrt{\Delta t}}{n\Delta t}}$$

$$= e^{-m^2/n},$$

where in the exponents we use

$$(n^2 - m^2)\Delta t \approx \infty, \qquad \frac{n\Delta t}{(n^2 - m^2)(\Delta t)^2} < \infty,$$

$$(n + m)\sqrt{\Delta t} \approx \infty \qquad \text{and} \quad \frac{m\sqrt{\Delta t}}{(n + m)\Delta t} < \infty.$$

So (3.3) is infinitely close to

$$\frac{1}{2\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{\sqrt{n}} \left(\frac{2m+1}{n+m+1}\right) e^{-\frac{m^2}{n}} \left(4p\,(1-p)\right)^n \left(\frac{1-p}{p}\right)^m. \tag{3.4}$$

On the other hand, from the power series expansion, we get

$$\begin{aligned} N \ln\left(4p(1-p)\right) &= N \ln\left(4\left(\frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right)\left(\frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right)\right) \\ &= -\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 + c\,\Delta t, \end{aligned}$$

for some finite c. That is

$$4p(1-p) = e^{-(\frac{r}{\sigma} - \frac{\sigma}{2})^2 \Delta t + c\,\Delta t^2}.$$

Raise both side of the equation to the power of n, we have

$$(4\,p\,(1-p))^n = \exp\left[-\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 n\Delta t + c(n\Delta t^2)\right].$$

Therefore (3.4) is infinitely close to

$$\frac{1}{2\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{\sqrt{n}} \left(\frac{2m+1}{n+m+1}\right) \exp\left[-\frac{m^2}{n} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 n\Delta t\right] \left(\frac{1-p}{p}\right)^m. \tag{3.5}$$

We have also for some finite c that

$$\frac{1-p}{p} = 1 + \left(\sigma - \frac{2r}{\sigma}\right)\sqrt{\Delta t} + c\Delta t,$$

and we can get from this that

$$\left(\frac{1-p}{p}\right)^m \approx e^{\left(\sigma - \frac{2r}{\sigma}\right) m\sqrt{\Delta t}}.$$

Therefore (3.5) is infinitely close to

$$\begin{aligned} &\frac{1}{2\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{\sqrt{n}\,\Delta t} \left(\frac{2m+1}{n+m+1}\right) \\ &\quad \times \exp\left[-\frac{m^2}{n} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 n\Delta t + \left(\sigma - \frac{2r}{\sigma}\right) m\sqrt{\Delta t}\right] \Delta t. \end{aligned} \tag{3.6}$$

Write

$$\alpha := m\sqrt{\Delta t} = -\frac{\ln L}{2\sigma},$$

then

$$\frac{F(2n\Delta t)}{\sqrt{n}\,\Delta t}\left(\frac{2m+1}{n+m+1}\right) = \left(\frac{F(2n\Delta t)}{(n+m+1)\Delta t}\right)\left(\frac{2m\sqrt{\Delta t}+\sqrt{\Delta t}}{\sqrt{n\Delta t}}\right)$$
$$\approx \frac{F(2n\Delta t)}{n\Delta t}\,\frac{2\alpha}{\sqrt{n\Delta t}}.$$

Therefore (3.6) is infinitely close to

$$\frac{1}{\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{n\Delta t}\,\frac{\alpha}{\sqrt{n\Delta t}} \times \exp\left[-\frac{\alpha^2}{n\Delta t} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 n\Delta t + \left(\sigma - \frac{2r}{\sigma}\right)\alpha\right]\Delta t, \tag{3.7}$$

i.e.,

$$\frac{e^{\left(\sigma - \frac{2r}{\sigma}\right)\alpha}}{\sqrt{\pi}} \sum_{M \le n < \frac{N}{2}} \frac{F(2n\Delta t)}{2n\Delta t}\,\frac{\sqrt{2}\alpha}{\sqrt{2n\Delta t}} \exp\left[-\frac{2\alpha^2}{2n\Delta t} - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 (2n\Delta t)\right](2\Delta t). \tag{3.8}$$

Viewed as a hyperfinite Riemann sum, (3.8) is infinitely close to

$$\frac{e^{\left(\sigma - \frac{2r}{\sigma}\right)\alpha}}{\sqrt{\pi}} \int_0^1 \frac{f(t)}{t}\,\frac{\sqrt{2}\alpha}{\sqrt{t}} \exp\left[-\frac{2\alpha^2}{t} - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 t\right] dt, \tag{3.9}$$

or, by using $e^{\alpha} = L^{-1/2\sigma}$,

$$\sqrt{\frac{2}{\pi}}\,\alpha\, L^{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)} \int_0^1 \frac{f(t)}{t} \exp\left[-\frac{2\alpha^2}{t} - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 t\right] \frac{dt}{\sqrt{t}}, \tag{3.10}$$

which is the following, by a change of variable,

$$2\sqrt{\frac{2}{\pi}}\,\alpha\, L^{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)} \int_0^1 \frac{f(t^2)}{t^2} \exp\left[-\frac{2\alpha^2}{t^2} - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 t^2\right] dt, \tag{3.11}$$

then the conclusion follows from the definition of the constants A, B, C. □

We denote the accumulated normal distribution function by

$$\mathcal{N}(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}\, du.$$

Theorem 3.1. *Consider margin trading on a risky asset whose volatility is* $\sigma > 0$ *and expected return rate is* $r \ge 0$, *both are standard. Set the margin call level to a standard number* $L \in (0, 1]$, *as a percentage of the present asset price.*

Then the expected margin call time within the standard time interval $[0, T]$ *is given by*

$$\begin{cases} \dfrac{\upsilon}{\sigma\,\kappa}\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)T & \text{if } r \neq \sigma^2/2 \text{ (i.e., } \kappa \neq 0) \\ -\left(\dfrac{\upsilon}{\sqrt{2\pi}}\,\dfrac{e^{-\frac{\upsilon^2}{2}}}{\mathcal{N}(\upsilon)} + \upsilon^2\right)T & \text{if } r = \sigma^2/2 \end{cases},$$

$$\text{where} \quad \begin{cases} \alpha = L^{-\kappa}\,\mathcal{N}(\upsilon - \sigma\,\kappa) \\ \beta = L^{\kappa}\,\mathcal{N}(\upsilon + \sigma\,\kappa) \\ \upsilon = \sigma^{-1}\,\ln L \\ \kappa = |r\sigma^{-2} - 2^{-1}|. \end{cases}$$

Proof. By normalizing the time interval $[0, T]$ to a unit time interval and using the present asset as the numéraire, we only need to compute the standard part of the expected margin call time given by the conditional expectation

$$\frac{\sum_{t\in\mathbb{T}} \mathbb{E}\left[(t+\Delta t)\,\mathbf{1}_{MC}(\omega,t)\right]}{\sum_{t\in\mathbb{T}} \mathbb{E}\left[\,\mathbf{1}_{MC}(\omega,t)\right]}. \tag{3.12}$$

Since $u^{-2(m+1)} \leq L \leq u^{-2m}$ for some $m \in {}^*\mathbb{N}$, and the standard part given by Lemma 3.1 is the same for either $L = u^{-2(m+1)}$ or $L = u^{-2m}$ we can work with an internal L of the form u^{-2m}, i.e., $\frac{-\ln L}{2\sigma\sqrt{\Delta t}} \in {}^*\mathbb{N}$, and apply Lemma 3.1 to $F(t) \equiv t + \Delta t$ and $F(t) \equiv 1$ separately.

First consider the case $\kappa \neq 0$.

For $F(t) \equiv t + \Delta t$, we have $f({}^\circ t) = {}^\circ F(t) = {}^\circ t$, and Lemma 3.1 gives

$$\sum_{t\in\mathbb{T}} \mathbb{E}\left[(t+\Delta t)\,\mathbf{1}_{MC}(\omega,t)\right] \approx C\int_0^1 e^{-\frac{A}{t^2}-Bt^2}\,dt,$$

where A, B, C are as in the lemma. From the integral

$$\int_0^1 e^{-\frac{A}{t^2}-Bt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{B}}\left(e^{-2\sqrt{AB}}\,\mathcal{N}\left(-\sqrt{2A}+\sqrt{2B}\right) - e^{2\sqrt{AB}}\,\mathcal{N}\left(-\sqrt{2A}-\sqrt{2B}\right)\right),$$

and

$$\sqrt{2A} = -\upsilon, \quad \sqrt{2B} = \sigma\,\kappa, \quad \text{and} \quad e^{2\sqrt{AB}} = L^{-\kappa}, \tag{3.13}$$

(note that $L \leq 1$ and hence $\upsilon \leq 0$) we obtain

$$\sum_{t\in\mathbb{T}} \mathbb{E}\left[(t+\Delta t)\,\mathbf{1}_{MC}(\omega,t)\right] \approx \frac{\upsilon}{\sigma\,\kappa}\,L^{\frac{r}{\sigma^2}-\frac{1}{2}}\left(\alpha-\beta\right). \tag{3.14}$$

Next for $F(t) \equiv 1$ we have $f(^\circ t) = {}^\circ F(t) = 1$, and Lemma 3.1 gives

$$\sum_{t \in \mathbb{T}} \mathbb{E}\left[\mathbf{1}_{MC}(\omega, t)\right] \approx C \int_0^1 \frac{1}{t^2}\, e^{-\frac{A}{t^2} - Bt^2}\, dt.$$

From

$$\int_0^1 \frac{1}{t^2}\, e^{-\frac{A}{t^2} - Bt^2}\, dt = \frac{1}{2}\sqrt{\frac{\pi}{A}} \left(e^{-2\sqrt{AB}}\, \mathcal{N}\left(-\sqrt{2A} + \sqrt{2B}\right) + e^{2\sqrt{AB}}\, \mathcal{N}\left(-\sqrt{2A} - \sqrt{2B}\right)\right),$$

and (3.13) we obtain

$$\sum_{t \in \mathbb{T}} \mathbb{E}\left[\mathbf{1}_{MC}(\omega, t)\right] \approx L^{\frac{r}{\sigma^2} - \frac{1}{2}} \left(\alpha + \beta\right), \tag{3.15}$$

and therefore the conclusion for the case $\kappa \neq 0$ follows.

For the other case, one obtains the formula by either applying Lemma 3.1 with $B = 0$ or letting $\kappa \to 0$ and get

$$\lim_{\kappa \to 0} \frac{\upsilon}{\sigma \kappa} \left(\frac{L^{-\kappa}\, \mathcal{N}(\upsilon - \sigma\kappa) - L^{\kappa}\, \mathcal{N}(\upsilon + \sigma\kappa)}{L^{-\kappa}\, \mathcal{N}(\upsilon - \sigma\kappa) + L^{\kappa}\, \mathcal{N}(\upsilon + \sigma\kappa)}\right) = -\left(\frac{\upsilon}{\sqrt{2\pi}}\, \frac{e^{-\frac{\upsilon^2}{2}}}{\mathcal{N}(\upsilon)} + \upsilon^2\right). \qquad \square$$

In applications, the expected margin call time given by Theorem 3.1 should serve as a measure for comparing risks in margin trading of various volatility and leverage structures.

Example 3.1. Let $T = 1$ and $r = 7\%$. Then we have the following expected margin call time for various σ and L in Table 1.

One can see as expected that margin call comes sooner for higher volatility and tighter margin call level.

4. Expected Margin Trading Loss and Its Maximum Level

In this section we calculate the expected loss from the margin trading and also study the leverage level that maximizes the loss for a given fixed volatility σ of the asset, i.e., the most dangerous level the trader must avoid.

Table 1. Expected call time in Example 3.1.

$L \setminus \sigma$	10%	15%	20%	25%	30%	35%	40%
95%	0.314	0.230	0.181	0.149	0.126	0.109	0.097
90%	0.530	0.410	0.332	0.279	0.240	0.210	0.187
85%	0.675	0.547	0.457	0.391	0.341	0.302	0.271
80%	0.772	0.652	0.559	0.488	0.431	0.386	0.349
70%	0.881	0.792	0.711	0.641	0.582	0.531	0.488
60%	0.934	0.874	0.812	0.752	0.698	0.649	0.606
50%	0.958	0.923	0.878	0.831	0.786	0.744	0.704

We still use time interval $[0, T]$, so each time $t \in \mathbb{T}$ corresponds to time ${}^\circ Tt$ in $[0, T]$. Let ρ be the fixed riskless interest rate which is the same as r in a risk-neutral environment. If margin call occurs at time $t \in \mathbb{T}$, the loss $(1-L)S_0$ would be incurred and equals $(1-L)S_0\, e^{-\rho Tt}$ when discounted back to the present. Therefore the expected total loss from margin calls is

$$(1-L)S_0 \sum_{t \in \mathbb{T}} \mathbb{E}\left[e^{-\rho T(t+\Delta t)}\, \mathbf{1}_{MC}(\omega, t)\right]. \tag{4.1}$$

By Lemma 3.1, (4.1) has standard part

$$(1-L)S_0\, C \int_0^1 \frac{e^{-\rho T t^2}}{t^2}\, e^{-\frac{A}{t^2} - Bt^2}\, dt, \tag{4.2}$$

which equals

$$\begin{aligned} &\frac{(1-L)S_0\, C}{2} \sqrt{\frac{\pi}{A}} \left(e^{2\sqrt{A(B+\rho T)}}\, \mathcal{N}\left(-\frac{\sqrt{2A}}{t} - \sqrt{2(B+\rho T)}\, t \right) \right. \\ &\qquad \left. - \, e^{-2\sqrt{A(B+\rho T)}}\, \mathcal{N}\left(\frac{\sqrt{2A}}{t} - \sqrt{2(B+\rho T)}\, t \right) \right). \end{aligned} \tag{4.3}$$

Upon simplification, we proved the following:

Theorem 4.1. *In a margin trading on a risky asset whose volatility is $\sigma > 0$, expected return rate is $r \geq 0$ and the present values is S_0, with margin call level $L \in (0, 1]$, riskless interest rate ρ, all numbers are standard, the expected loss due to margin calls during the time interval $[0, T]$ is given by*

$$S_0\, L^{\theta/\sigma}\, (1-L)\, \left(L^{\xi}\, \mathcal{N}\left(\sigma^{-1} \ln L + \sigma\,\xi\right) \,+\, L^{-\xi}\, \mathcal{N}\left(\sigma^{-1} \ln L - \sigma\,\xi\right) \right),$$
$$\text{where} \quad \begin{cases} \theta = r/\sigma - \sigma/2 \\ \xi = \sqrt{\theta^2 + 2\rho T}\, /\, \sigma \end{cases}. \tag{4.4}$$

Further calculations and simplification gives the following which is needed in considering the sensitivity of the loss to the margin level L:

Corollary 4.1. *The derivative of the expected loss with respect to L is*

$$L^{\frac{r}{\sigma^2} - \frac{3}{2}}\, (1-L)\, S_0 \left(\eta(\nu) + \eta(-\nu) + \frac{1}{\sigma} \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}\left(\nu^2 + \left(\frac{\ln L}{\sigma}\right)^2\right)} \right),$$
$$\text{where} \quad \begin{cases} \nu = \sqrt{2\, T\rho + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2} \\ \eta(x) = L^{x/\sigma}\, \left(\frac{L}{L-1} + \frac{r}{\sigma^2} + \frac{x}{\sigma} - \frac{1}{2} \right) \mathcal{N}\left(x + \frac{\ln L}{\sigma}\right) \end{cases}.$$

Note that the expected margin loss is 0 at $L = 1$ and approaches to 0 as $L \to 0^+$, with the former essentially representing no margin trading (the expected margin call time is 0 in such case) and the latter representing setting the margin call level as low as possible. The formula in the above corollary enables one to calculate the

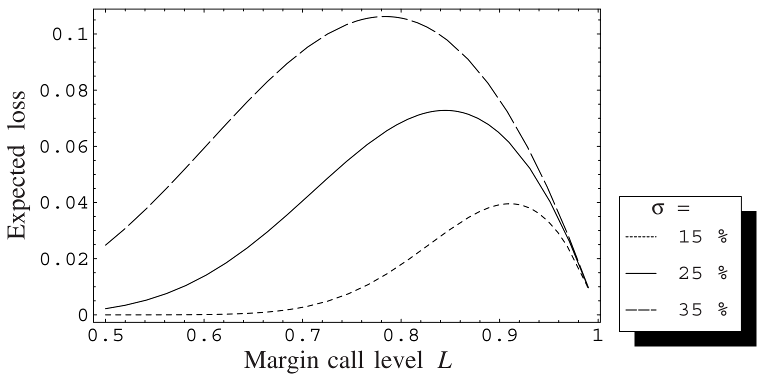

Fig. 1. Expected loss versus call level in Example 4.1.

L with all other parameters fixed, for which maximum loss takes place; this is of course the dangerous level any trader should avoid.

Example 4.1. Let $r = 5\%$, $\rho = 3\%$, $S_0 = 1$ and $T = 1$. Figure 1 shows the relationship between the expected margin loss and various margin call levels L when the volatility σ is at 15%, 25% and 35% respectively.

Applying the formula in Corollary 4.1, one can solve numerically the level L for which maximum expected loss occurs: $L = 0.91029$ for $\sigma = 15\%$, $L = 0.84477$ for $\sigma = 25\%$ and $L = 0.78374$ for $\sigma = 35\%$.

5. A Persistent Strategy

We now describe a trading strategy in which the trader persists in the margin trading immediately after it was called with whatever margin left. Surprisingly the additional risk does not seem to be as great as one might imagine.

As in the Introduction, let $\gamma \in (0, 1)$ be the margin requirement, P the margin, i.e., margin call occurs when the asset value is below $S_0 - \gamma P$. Also the margin call level L is $1 - \gamma/\lambda$, where λ is the leverage and $P = \frac{(1-L)\, S_0}{\gamma}$.

Then, if on path ω a margin is called at time $t + \Delta t$, $(1 - L)\, S_0$ is lost at that point, the trader is left with $(1-\gamma)\, P$. In the persistent strategy, the trader continue the margin trading immediately at time $t + \Delta t$ with the new margin $(1 - \gamma)\, P$ and the same leverage λ hence the same L. The trader's new position at the margin call time $t + \Delta t$ is therefore $(1 - \gamma)\, S_0$.

Let us call this 1-persistent strategy, and an n-persistent strategy is a repetition of the above n-times.

Let $\phi_n(x, \tau)$ denote the expected loss in an n-persistent strategy with present position x and time horizon $[0, \tau]$. For convenience, we regard the plain margin trading as the 0-persistent strategy, so $\phi_0(S_0, T)$ is given by formula (4.4).

Theorem 5.1. *Under the same assumptions in Theorem* 4.1, *let* $\phi_0(S_0, T)$ *be given by formula* (4.4). *Let* $\gamma \in (0, 1)$ *be the margin requirement.*

Then the expected loss in an n-persistent strategy is given recursively by

$$\phi_{n+1}(S_0,\,T) = \phi_0(S_0,\,T) + \int_0^1 \frac{C}{t^2}\,\left(\phi_n\left((1-\gamma)S_0,\,(1-t^2)\,T\right)\right)\,e^{-\frac{A}{t^2}-(B+\rho T)t^2}dt, \tag{5.1}$$

where A, B, C are as in Lemma 3.1.

Proof. We simply note that the additional loss from the $(n+1)$-persistent strategy $\phi_{n+1}(S_0,\,T) - \phi_0(S_0,\,T)$ is the standard part of

$$\sum_{t\in\mathbb{T}} \mathbb{E}\left[e^{-\rho T(t+\Delta t)}\,\left(\phi_n\left((1-\gamma)S_0,\,(1-t)\,T\right)\right)\,1_{MC}\,(\omega,t)\right],$$

then apply Lemma 3.1. □

At first it seems that, at least in the case $\rho = 0$, keep repeating the persistent strategy may lead to the complete loss of the initial margin $\frac{(1-L)\,S_0}{\gamma}$. But the following example suggests that this is not the case.

Example 5.1. Let $\sigma = 25\%$, $r = 5\%$, $\rho = 0\%$, $\gamma = 0.5$, $S_0 = 1$ and $T = 1$. Figure 2 shows the increasing loss due to the n-persistent strategy for $n = 0, 1, 2, 3, 4, 5, 10, 15$ at various margin call levels L. Note that the last three graphs are hardly distinguishable, suggesting rather fast convergence rate as $n \to \infty$. Moreover, even at the level L corresponding to maximum loss, the expected loss is capped by $(1-L)S_0$. Also the small additional loss due to any n-persistent strategy indicates that instead of a plain margin trading the persistent strategy is a sensible one and is worth pursuing.

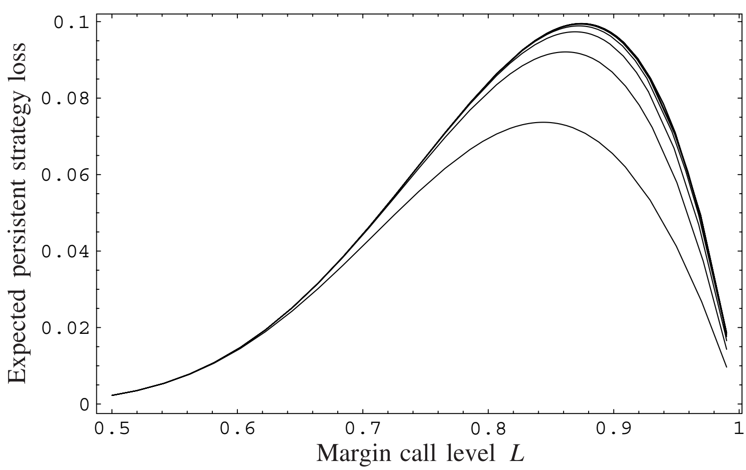

Fig. 2. Expected loss from the persistent strategy in Example 5.1.

Note that $\phi_n(S_0, T)$ is increasing in n and we define the limit of the persistent strategy as:

$$\phi_\infty(S_0, T) = \lim_{n\to\infty} \phi_n(S_0, T).$$

We conjecture the following between the plain margin trading and the persistent strategy:

$$\phi_\infty(S_0, T) \leq \gamma^{-1}\, \phi_0(S_0, T). \tag{5.2}$$

6. Application to Pricing a Barrier Option

In this section we apply the formula from Lemma 3.1 to pricing a particular kind of exotic option, namely the *down knock-in barrier call option*. With some modification the same methodology in this paper is also applicable to other styles of barrier options: any combination of down/up, knock-in/knock-out, call/put. See [9] for more details on a standard treatment of these and other related options.

For a down knock-in barrier call option to become active on the path ω, a prefixed barrier $L\, S_0$, where $0 < L \leq 1$ must be touched by $S_t(\omega)$ at some time $t \in [0, T]$; and if this is so, it will behave from that point on like an European call option with some strike price $K \geq 0$ and exercise time T. We assume a risk-neutral environment, i.e., the expected asset return rate r is the same as the riskless interest rate.

Now we can derive an explicit formula for pricing the option.

Theorem 6.1. *Consider a down knock-in barrier call option written on a risky asset with volatility $\sigma > 0$, expected return rate being the same as the riskless interest rate $r \geq 0$, present values S_0, where a lower barrier is fixed at $L\, S_0$, for some $0 < L \leq 1$, the strike price is K and the exercise time is T. The price of this option is given by*:

$$\int_0^1 \frac{\theta(t^2)}{t^2}\, e^{-\frac{A}{t^2} - Bt^2}\, dt, \tag{6.1}$$

where

$$\theta(t) = C\Bigg(L\, S_0\, e^{-rTt}\, \mathcal{N}\left(\frac{\ln(LS_0) - \ln K}{\sigma\sqrt{T(1-t)}} + \left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T(1-t)} \right) - K\, e^{-rT}\, \mathcal{N}\left(\frac{\ln(LS_0) - \ln K}{\sigma\sqrt{T(1-t)}} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T(1-t)} \right) \Bigg)$$

and A, B, C are as given in Lemma 3.1.

Proof. First note that the indicator function $\mathbf{1}_{MC}(\omega, t)$ is also that of first touching the level $L\, u^{-1}\, S_0 = L\, e^{-\sigma\sqrt{\Delta t}}\, S_0$ at time $t + \Delta t$.

As $L \approx L\, u^{-1}$, the result follows by applying Lemma 3.1 with $f(t) =$ the discounted Black–Scholes price at time t, i.e., with

$$f(t) = e^{-rTt}\,(\text{Black–Scholes price at time } t).$$

We then take $\theta(t) = C\, f(t)$. □

Remark 6.1. In [9], the image method from PDE is used (see also [6]) and an explicit pricing formula is also given. For the case $K > L\,S_0$ the formula there is equivalent to:

$$L^{\frac{2r}{\sigma^2}+1}\, S_0\, \mathcal{N}\left(\frac{\ln(L^2 S_0) - \ln K}{\sigma\sqrt{T}} + \left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T}\right)$$
$$- L^{\frac{2r}{\sigma^2}-1}\, K\, e^{-rT}\, \mathcal{N}\left(\frac{\ln(L^2 S_0) - \ln K}{\sigma\sqrt{T}} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}\right),$$

and this should be in agreement with (6.1).

Appendix

In this section, we prove a technical combinatorial result, namely a generalization of the Catalan numbers, that was used earlier in the proof of Lemma 3.1. See [8] for more details about this generalization. Other kinds of generalization as well as a comprehensive introduction to the Catalan numbers can be found in [4].

Here only sequences of the form $\{\omega_i\}_{j\le i\le k}$ where $j, k \in {}^*\mathbb{Z}$ and $\omega_i = \pm 1$ are considered. When applied to the sequence $\omega \in \Omega = \{-1,+1\}^{\mathbb{T}}$, $\omega_{n\Delta t}$ is identified with ω_n. We abbreviate the partial sum of the (± 1)'s as

$$\sigma_j^k(\omega) := \sum_{j\le i<k} \omega_i.$$

For $0 \le m \le n$ the number of sequences $\{\omega_i\}_{0\le i<2n}$ such that $\sigma_0^k(\omega) \ge -2m$, where $0 \le k < 2n$, and $\sigma_0^{2n}(\omega) = -2m$ is denoted by

$$\mathcal{C}_{n,m}.$$

Note that $\mathcal{C}_{n,0}$ corresponds to the classical Catalan numbers.

Theorem A.1. *Given integers $0 \le m \le n$, we have*

$$\mathcal{C}_{n,m} = \frac{2m+1}{n+m+1}\binom{2n}{n+m}. \tag{A.2}$$

Proof. Fix $0 \le m \le n$. We first define the following sets of sequences of (± 1)'s:

$$\Lambda := \left\{ \{\omega_i\}_{0\le i<2n} \mid \quad \sigma_0^{2n}(\omega) = -2m \right\},$$
$$\Gamma := \left\{ \omega \in \Lambda \mid \quad \sigma_0^k(\omega) < -2m \quad \text{for some} \quad k \right\},$$
$$\Xi := \left\{ \{\omega_i\}_{0\le i<2n} \mid \quad \sigma_0^{2n}(\omega) = 2(m+1) \right\}.$$

So $\mathcal{C}_{n,m} = |\Lambda| - |\Gamma|$. We first define a bijection between Γ and Ξ.

For $\omega \in \Gamma$ we can define

$$j(\omega) := \min\{k | \sigma_0^{k+1}(\omega) = -2m-1\}.$$

Now define $\theta : \Gamma \to \Xi$ by

$$\theta(\omega)_i = \begin{cases} -\omega_i & i \le j(\omega) \\ \omega_i & \text{otherwise} \end{cases}.$$

(This is called André's reflection method.) It is easy to see that θ is a bijection. Therefore we have

$$\mathcal{C}_{n,m} = |\Lambda| - |\Xi|.$$

On the other hand, $|\Lambda| = \binom{2n}{n+m}$, since each $\omega \in \Lambda$ consists of $n+m$ (-1)'s and $n-m$ $(+1)$'s; similarly $|\Xi| = \binom{2n}{n-m-1}$. Hence

$$\mathcal{C}_{n,m} = \binom{2n}{n+m} - \binom{2n}{n-m-1},$$

and the conclusion follows by noticing that

$$\binom{2n}{n-m-1} = \frac{n-m}{n+m+1}\binom{2n}{n-m} = \frac{n-m}{n+m+1}\binom{2n}{n+m}. \qquad \square$$

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